ACTA ARITHMETICA
* (201*)

# Number of solutions in a box of a linear homogeneous equation in an Abelian group

by

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**1. Introduction.** K. Cwalina and T. Schoen [1] have recently proved the following conjecture of A. Schinzel [3]: the number of solutions of the congruence $a_1x_1 + \cdots + a_kx_k \equiv 0 \pmod{n}$ in the box $0 \le x_i \le b_i$, where $b_i$ are positive integers, is at least $2^{1-n}\prod_{i=1}^{k}(b_i+1)$. Using a completely different method we shall prove the following more general statement, also conjectured by Schinzel ([3, p. 364]).

THEOREM 1.1. *For every finite Abelian group $\Gamma$, for all $a_1, \ldots, a_k \in \Gamma$, and for all positive integers $b_1, \ldots, b_k$ the number of solutions of the equation $\sum_{i=1}^{k} a_ix_i = 0$ in nonnegative integers $x_i \le b_i$ is at least*

$$2^{1-D(\Gamma)}\prod_{i=1}^{k}(b_i+1), \tag{1.1}$$

*where $D(\Gamma)$ is the Davenport constant of the group $\Gamma$ (see Def. 2.1 below).*

**2. Lemmas and definitions.** Let $\Gamma$ be a finite Abelian group, with multiplicative notation.

DEFINITION 2.1. Define the *Davenport constant* $D(\Gamma)$ to be the smallest positive integer $n$ such that, for any sequence $g_1, \ldots, g_n$ of group elements, there exist indices

$$1 \le i_1 < \cdots < i_t \le n \quad \text{for which} \quad g_{i_1} \cdot \ldots \cdot g_{i_t} = 1.$$

For a group with multiplicative notation, Theorem 1.1 has the form: for every finite Abelian group $\Gamma$, for all $a_1, \ldots, a_k \in \Gamma$, and for all positive integers $b_1, \ldots, b_k$ the number of solutions of the equation $\prod_{i=1}^{k} a_i^{x_i} = 1$ in

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nonnegative integers $x_i \le b_i$ is at least

$$2^{1-D(\mathit{\Gamma})}\prod_{i=1}^{k}(b_i+1). \tag{2.1}$$

By the definition of the Davenport constant for the group $\mathit{\Gamma}$, we may find $g_1,\dots,g_{D(\mathit{\Gamma})-1}\in\mathit{\Gamma}$ such that any product of a nonempty subsequence of this sequence is not equal to 1 in $\mathit{\Gamma}$.

Since the number of solutions of the equation $\prod_{i=1}^{D(\mathit{\Gamma})-1} g_i^{x_i}=1$, where $x_i=0$ or $x_i=1$, is equal to $1=2^{1-D(\mathit{\Gamma})}\prod_{i=1}^{D(\mathit{\Gamma})-1}(1+1)$, we obtain:

Remark 2.2. If Theorem 1.1 is true, then $2^{1-D(\mathit{\Gamma})}$ is the best possible coefficient independent of $a_i$, $b_i$ and depending only on $\mathit{\Gamma}$.

Lemma 2.3. *For $n\ge 1$ we have the following identity in $\mathbb{Q}[x]$ and in $\mathbb{Q}[\mathit{\Gamma}]$:*

$$1+x+x^2+\dots+x^n=\sum_{j=0}^{n}2^{j-n-1}(1+x^j)(1+x)^{n-j}. \tag{2.2}$$

*Proof.* We proceed by induction on $n$. For $n=1$ we have

$$\sum_{j=0}^{1}2^{j-1-1}(1+x^j)(1+x)^{1-j}=2^{-2}(1+1)(1+x)+2^{-1}(1+x)=1+x$$

and the assertion is true.

Assume it is true for degrees less than $n$, where $n>1$. Then

$$\begin{aligned}
1+x+x^2+\dots+x^n&=\frac{1}{2}\big((1+x)(1+x+\dots+x^{n-1})+(1+x^n)\big)\\
&=\frac{1}{2}\Big((1+x)\sum_{j=0}^{n-1}2^{j-(n-1)-1}(1+x^j)(1+x)^{n-1-j}+(1+x^n)\Big)\\
&=\sum_{j=0}^{n-1}2^{j-n-1}(1+x^j)(1+x)^{n-j}+\frac{1}{2}(1+x^n)\\
&=\sum_{j=0}^{n}2^{j-n-1}(1+x^j)(1+x)^{n-j}. \quad\blacksquare
\end{aligned}$$

Definition 2.4. For an element $\sum_{g\in\mathit{\Gamma}}N_g g$ of the group ring $\mathbb{Q}[\mathit{\Gamma}]$ and a number $n\in\mathbb{Q}$ we write

$$\sum_{g\in\mathit{\Gamma}}N_g g\succeq n \quad \text{iff} \quad N_1\ge n.$$

Lemma 2.5. *Theorem 1.1 in multiplicative notation is equivalent to the statement: for every finite Abelian group $\mathit{\Gamma}$, for all $a_1,\dots,a_k\in\mathit{\Gamma}$, and for* *all positive integers* $b_1, \dots, b_k$ *we have relation:*

$$\prod_{i=1}^{k}(1 + a_i + \dots + a_i^{b_i}) \succeq 2^{1-D(\Gamma)} \prod_{i=1}^{k}(b_i + 1), \tag{2.3}$$

*where* $D(\Gamma)$ *is the Davenport constant of the group* $\Gamma$*.*

*Proof.* Indeed, the number of solutions of the equation $\prod_{i=1}^{k} a_i^{x_i} = 1$ in nonnegative integers $x_i \le b_i$ is equal to $N_1$, where

$$\prod_{i=1}^{k}(1 + a_i + \dots + a_i^{b_i}) = \sum_{g\in\Gamma} N_g g.$$

We have $N_1 \ge 2^{1-D(\Gamma)} \prod_{i=1}^{k}(b_i + 1)$ if and only if relation (2.3) holds. ∎

LEMMA 2.6. *Let* $\Gamma$ *be a finite Abelian group. For all* $a_1, \dots, a_k \in \Gamma$ *we have*

$$(1 + a_1) \cdot \ldots \cdot (1 + a_k) \succeq 2^{1-D(\Gamma)} \cdot 2^k. \tag{2.4}$$

*Proof.* For the completeness of exposition we provide Olson's proof [2].

We proceed by induction on $k$. For $k \le D(\Gamma) - 1$ we have

$$(1 + a_1) \cdot \ldots \cdot (1 + a_k) \succeq 1 \ge 2^{1-D(\Gamma)} \cdot 2^k$$

and the assertion is true.

Assume it is true for the number of factors less than $k$, where $k > D(\Gamma) - 1$. Hence $k \ge D(\Gamma)$. By the definition of the Davenport constant we may assume, without loss of generality, that

$$a_1 \cdot \ldots \cdot a_t = 1 \quad \text{for some } 1 \le t \le D(\Gamma).$$

By the inductive assumption

$$\prod_{i=2}^{t}(1 + a_i^{-1}) \prod_{i=t+1}^{k}(1 + a_i) \succeq 2^{1-D(\Gamma)} \cdot 2^{k-1},$$
$$\prod_{i=2}^{k}(1 + a_i) \succeq 2^{1-D(\Gamma)} \cdot 2^{k-1}.$$

Hence

$$\begin{aligned}\prod_{i=1}^{k}(1+a_i) &= \prod_{i=2}^{k}(1+a_i) + a_1\prod_{i=2}^{k}(1+a_i)\\ &= \prod_{i=2}^{k}(1+a_i) + a_1\cdot\ldots\cdot a_t\prod_{i=2}^{t}(1+a_i^{-1})\prod_{i=t+1}^{k}(1+a_i)\\ &= \prod_{i=2}^{k}(1+a_i) + \prod_{i=2}^{t}(1+a_i^{-1})\prod_{i=t+1}^{k}(1+a_i)\\ &\succeq 2^{1-D(\Gamma)}\cdot 2^{k-1} + 2^{1-D(\Gamma)}\cdot 2^{k-1} = 2^{1-D(\Gamma)}\cdot 2^{k}. \ \blacksquare\end{aligned}$$

**3. Proof of Theorem.** By Lemma 2.5 it suffices to prove:

THEOREM. *For every finite Abelian group $\Gamma$, for all $a_1,\ldots,a_k \in \Gamma$, and for all positive integers $b_1,\ldots,b_k$ we have*

$$\prod_{i=1}^{k}(1+a_i+\cdots+a_i^{b_i}) \succeq 2^{1-D(\Gamma)}\prod_{i=1}^{k}(b_i+1),$$

*where $D(\Gamma)$ is the Davenport constant of the group $\Gamma$.*

*Proof.* We use the identity (2.2) to get

$$P(a_1,\ldots,a_k) = \prod_{i=1}^{k}(1+a_i+\cdots+a_i^{b_i}) = \prod_{i=1}^{k}\sum_{j=0}^{b_i}2^{j-b_i-1}(1+a_i^{j})(1+a_i)^{b_i-j}.$$

Hence for a certain $s$ we obtain $P(a_1,\ldots,a_k) = \sum_{1\le i\le s} v_i P_i(a_1,\ldots,a_k)$, where $v_i$ are positive rational numbers and each $P_i(a_1,\ldots,a_k)$ has the form

$$(1+c_1)\cdot\ldots\cdot(1+c_m),$$

where $c_1,\ldots,c_m \in \Gamma$.

For $P_i(a_1,\ldots,a_k)$ we use Lemma 2.6 to get

$$P_i(a_1,\ldots,a_k) \succeq 2^{1-D(\Gamma)}P_i(1,\ldots,1), \quad 1\le i\le s.$$

Note that we use $P$, $P_i$ in two different domains at the same time, in $\mathbb{Q}[\Gamma]$ and in $\mathbb{Q}[x]$.

It follows that $P(a_1,\ldots,a_k) \succeq 2^{1-D(\Gamma)}P(1,\ldots,1)$. Thus

$$\prod_{i=1}^{k}(1+a_i+\cdots+a_i^{b_i}) \succeq 2^{1-D(\Gamma)}\prod_{i=1}^{k}(b_i+1). \ \blacksquare$$

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## References

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