Number of solutions in a box of a linear homogeneous equation in an Abelian group

by

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1. Introduction. K. Cwalina and T. Schoen [1] have recently proved the following conjecture of A. Schinzel [3]: the number of solutions of the congruence

\[ a_1 x_1 + \cdots + a_k x_k \equiv 0 \pmod{n} \]

in the box \(0 \leq x_i \leq b_i\), where \(b_i\) are positive integers, is at least \(2^{1-n} \prod_{i=1}^{k} (b_i + 1)\). Using a completely different method we shall prove the following more general statement, also conjectured by Schinzel ([3, p. 364]).

Theorem 1.1. For every finite Abelian group \(\Gamma\), for all \(a_1, \ldots, a_k \in \Gamma\), and for all positive integers \(b_1, \ldots, b_k\) the number of solutions of the equation

\[ \sum_{i=1}^{k} a_i x_i = 0 \]

in nonnegative integers \(x_i \leq b_i\) is at least

\[ 2^{1-D(\Gamma)} \prod_{i=1}^{k} (b_i + 1), \]

(1.1)

where \(D(\Gamma)\) is the Davenport constant of the group \(\Gamma\) (see Def. 2.1 below).

2. Lemmas and definitions. Let \(\Gamma\) be a finite Abelian group, with multiplicative notation.

Definition 2.1. Define the Davenport constant \(D(\Gamma)\) to be the smallest positive integer \(n\) such that, for any sequence \(g_1, \ldots, g_n\) of group elements, there exist indices

\[ 1 \leq i_1 < \cdots < i_t \leq n \]

for which \(g_{i_1} \cdots g_{i_t} = 1\).

For a group with multiplicative notation, Theorem [1.1] has the form: for every finite Abelian group \(\Gamma\), for all \(a_1, \ldots, a_k \in \Gamma\), and for positive integers \(b_1, \ldots, b_k\) the number of solutions of the equation \(\prod_{i=1}^{k} a_i^{x_i} = 1\) in

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nonnegative integers \( x_i \leq b_i \) is at least

\[
2^{1-D(\Gamma)} \prod_{i=1}^{k} (b_i + 1).
\]

By the definition of the Davenport constant for the group \( \Gamma \), we may find \( g_1, \ldots, g_{D(\Gamma)-1} \in \Gamma \) such that any product of a nonempty subsequence of this sequence is not equal to 1 in \( \Gamma \).

Since the number of solutions of the equation \( \prod_{i=1}^{D(\Gamma)-1} g_i^{x_i} = 1 \), where \( x_i = 0 \) or \( x_i = 1 \), is equal to \( 1 = 2^{1-D(\Gamma)} \prod_{i=1}^{D(\Gamma)-1} (1 + 1) \), we obtain:

**Remark 2.2.** If Theorem 1.1 is true, then \( 2^{1-D(\Gamma)} \) is the best possible coefficient independent of \( a_i, b_i \) and depending only on \( \Gamma \).

**Lemma 2.3.** For \( n \geq 1 \) we have the following identity in \( \mathbb{Q}[x] \) and in \( \mathbb{Q}[\Gamma] \):

\[
1 + x + x^2 + \cdots + x^n = \sum_{j=0}^{n} 2^{j-n-1} (1 + x^j)(1 + x)^{n-j}.
\]

**Proof.** We proceed by induction on \( n \). For \( n = 1 \) we have

\[
\sum_{j=0}^{1} 2^{j-n-1} (1 + x^j)(1 + x)^{1-j} = 2^{-2}(1 + 1)(1 + x) + 2^{-1}(1 + x) = 1 + x
\]

and the assertion is true.

Assume it is true for degrees less than \( n \), where \( n > 1 \). Then

\[
1 + x + x^2 + \cdots + x^n = \frac{1}{2}((1 + x)(1 + x + \cdots + x^{n-1}) + (1 + x^n))
\]

\[
= \frac{1}{2} \left((1 + x) \sum_{j=0}^{n-1} 2^{j-(n-1)-1} (1 + x^j)(1 + x)^{n-1-j} + (1 + x^n)\right)
\]

\[
= \sum_{j=0}^{n-1} 2^{j-n-1} (1 + x^j)(1 + x)^{n-j} + \frac{1}{2} (1 + x^n)
\]

\[
= \sum_{j=0}^{n} 2^{j-n-1} (1 + x^j)(1 + x)^{n-j}.
\]

**Definition 2.4.** For an element \( \sum_{g \in \Gamma} N_g g \) of the group ring \( \mathbb{Q}[\Gamma] \) and a number \( n \in \mathbb{Q} \) we write

\[
\sum_{g \in \Gamma} N_g g \geq n \quad \text{iff} \quad N_1 \geq n.
\]

**Lemma 2.5.** Theorem 1.1 in multiplicative notation is equivalent to the statement: for every finite Abelian group \( \Gamma \), for all \( a_1, \ldots, a_k \in \Gamma \), and for
all positive integers $b_1, \ldots, b_k$ we have relation:

\[
(2.3) \quad \prod_{i=1}^{k} (1 + a_i + \cdots + a_i^{b_i}) \geq 2^{1-D(\Gamma)} \prod_{i=1}^{k} (b_i + 1),
\]

where $D(\Gamma)$ is the Davenport constant of the group $\Gamma$.

**Proof.** Indeed, the number of solutions of the equation $\prod_{i=1}^{k} a_i^{x_i} = 1$ in nonnegative integers $x_i \leq b_i$ is equal to $N_1$, where

\[
\prod_{i=1}^{k} (1 + a_i + \cdots + a_i^{b_i}) = \sum_{g \in \Gamma} N_g g.
\]

We have $N_1 \geq 2^{1-D(\Gamma)} \prod_{i=1}^{k} (b_i + 1)$ if and only if relation (2.3) holds. ■

**Lemma 2.6.** Let $\Gamma$ be a finite Abelian group. For all $a_1, \ldots, a_k \in \Gamma$ we have

\[
(2.4) \quad (1 + a_1) \cdots (1 + a_k) \geq 2^{1-D(\Gamma)} \cdot 2^k.
\]

**Proof.** For the completeness of exposition we provide Olson’s proof [2].

We proceed by induction on $k$. For $k \leq D(\Gamma) - 1$ we have

\[
(1 + a_1) \cdots (1 + a_k) \geq 1 \geq 2^{1-D(\Gamma)} \cdot 2^k
\]

and the assertion is true.

Assume it is true for the number of factors less than $k$, where $k > D(\Gamma) - 1$. Hence $k \geq D(\Gamma)$. By the definition of the Davenport constant we may assume, without loss of generality, that

\[
a_1 \cdots a_t = 1 \quad \text{for some } 1 \leq t \leq D(\Gamma).
\]

By the inductive assumption

\[
\prod_{i=2}^{t} (1 + a_i^{-1}) \prod_{i=t+1}^{k} (1 + a_i) \geq 2^{1-D(\Gamma)} \cdot 2^{k-1},
\]

\[
\prod_{i=2}^{k} (1 + a_i) \geq 2^{1-D(\Gamma)} \cdot 2^{k-1}.
\]
Hence
\[
\prod_{i=1}^{k}(1 + a_i) = \prod_{i=2}^{k}(1 + a_i) + a_1 \prod_{i=2}^{k}(1 + a_i)
\]
\[
= \prod_{i=2}^{k}(1 + a_i) + a_1 \cdot \ldots \cdot a_t \prod_{i=2}^{t}(1 + a_i^{-1}) \prod_{i=t+1}^{k}(1 + a_i)
\]
\[
= \prod_{i=2}^{k}(1 + a_i) + \prod_{i=2}^{t}(1 + a_i^{-1}) \prod_{i=t+1}^{k}(1 + a_i)
\]
\[
\geq 2^{1-D(\Gamma)} \cdot 2^{k-1} + 2^{1-D(\Gamma)} \cdot 2^{k-1} = 2^{1-D(\Gamma)} \cdot 2^k. \]

3. Proof of Theorem. By Lemma 2.5 it suffices to prove:

**Theorem.** For every finite Abelian group \(\Gamma\), for all \(a_1, \ldots, a_k \in \Gamma\), and for all positive integers \(b_1, \ldots, b_k\) we have
\[
\prod_{i=1}^{k}(1 + a_i + \cdots + a_i^{b_i}) \geq 2^{1-D(\Gamma)} \prod_{i=1}^{k}(b_i + 1),
\]
where \(D(\Gamma)\) is the Davenport constant of the group \(\Gamma\).

**Proof.** We use the identity (2.2) to get
\[
P(a_1, \ldots, a_k) = \prod_{i=1}^{k}(1 + a_i + \cdots + a_i^{b_i}) = \prod_{i=1}^{k} \sum_{j=0}^{b_i} 2^{j-b_i-1}(1 + a_i^j)(1 + a_i)^{b_i-j}.
\]
Hence for a certain \(s\) we obtain
\[
P(a_1, \ldots, a_k) = \sum_{1 \leq i \leq s} v_i P_i(a_1, \ldots, a_k),
\]
where \(v_i\) are positive rational numbers and each \(P_i(a_1, \ldots, a_k)\) has the form
\[
(1 + c_1) \cdot \ldots \cdot (1 + c_m),
\]
where \(c_1, \ldots, c_m \in \Gamma\).

For \(P_i(a_1, \ldots, a_k)\) we use Lemma 2.6 to get
\[
P_i(a_1, \ldots, a_k) \geq 2^{1-D(\Gamma)} P_i(1, \ldots, 1), \quad 1 \leq i \leq s.
\]
Note that we use \(P, P_i\) in two different domains at the same time, in \(\mathbb{Q}[\Gamma]\) and in \(\mathbb{Q}[x]\).

It follows that
\[
P(a_1, \ldots, a_k) \geq 2^{1-D(\Gamma)} P(1, \ldots, 1). \quad \text{Thus}
\]
\[
\prod_{i=1}^{k}(1 + a_i + \cdots + a_i^{b_i}) \geq 2^{1-D(\Gamma)} \prod_{i=1}^{k}(b_i + 1). \]

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