.

On some cancellation algorithms

Maciej Zakarczemny

Cracow University of Technology, Poland

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Assume that $g : \mathbb{N} \to \mathbb{N}$ is **injective** mapping. Let:

$$D_g(n) := \min\{m \in \mathbb{N} : g(1), g(2), \ldots, g(n) \text{ are distinct modulo } m\}. \qquad (1)$$

The function D_g is commonly called the discriminator of the function g, because it provides the least modulus which discriminates the successive values of the function g.

The problem first appears in the context of the computation of square roots of a long sequence of integers.

Arnold, Benkoski, and McCabe [1] defined, for a natural number n, the smallest natural number m such that $1^2, 2^2, \ldots, n^2$ are all distinct modulo m.

In this case, the value $D_g(n)$ for $n > 4$ is the smallest $m \geq 2n$ such that m is a prime or twice a prime.

Bremser, Schumer, Washington [2] determined for each sufficiently large natural number, the smallest positive integer m such that $1^j, 2^j, \ldots, n^j$ are all incongruent modulo m.

[1] L.K. Arnold, S.J. Benkoski and B.J. McCabe, *The discriminator (a simple application of Bertrand's postulate)*, Amer. Math. Monthly (1985), 92, 275-277.

[2] P. S. Bremser, P.D. Schumer, L.C. Washington, *A note on the incongruence of consecutive integers to a fixed power*, J. Number Theory (1990), 35, no. 1, 105-108.

Lately, the discriminators of various types of functions were considered by Zieve [12], Sun [8], Moree and Zumalacárrequi [6], Haque and Shallit [4].

[4] S. Haque and J. Shallit *Discriminators and k-regular sequences* INTEGERS 16(2106), Paper A76.

[6] P. Moree and A. Zumalacárregui, *Salajan's conjecture on discriminating terms in an exponential sequence*, J. Number Theory 160(2016),646-665.

[8] Zhi-Wei Sun, *On funtions taking only prime values*, J. Number Theory 133(2013), 2794-2812.

[12] M. Zieve, *A note on the discriminator*, J. Number Theory 73(1998), 122-138.

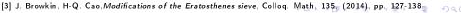There is also a slightly different definition of a discriminator in terms of cancellations algorithms.

Instead of (1) Browkin and Cao also considered an arbitrary function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with the set

$$A_n = \{f(n_1, n_2) : n_1 + n_2 \leq n;\ n_1, n_2 \in \mathbb{N}\}.$$

They cancel from $\mathbb{N}$ all numbers from the set of divisors

$$D_n = \{d \in \mathbb{N} : d|m \text{ for some } m \in A_n\},$$

and define $b_f(n)$ to be the least non-canceled number (see Browkin and Cao in the paper [3]).

[3] J. Browkin, H-Q. Cao, *Modifications of the Eratosthenes sieve*, Colloq. Math. 135 (2014), pp. 127-138

Example

If

$$A_n = \{(n_1 + n_2)^2 - n_2^2 : n_1 + n_2 \leq n;\ n_1, n_2 \in \mathbb{N}\}.$$

$$D_n = \{d \in \mathbb{N} : \exists_{n_1, n_2 \in \mathbb{N},\ n_1 + n_2 \leq n}\ d | (n_1 + n_2)^2 - n_2^2\}$$

and $b_f(n)$ is the least number in the set $\mathbb{N} \setminus D_n$ then

$A_1 = \emptyset$	
$D_1 = \emptyset$	$b_f(1) = 1,$
$A_2 = \{3\}$	
$D_2 = \{1, 3\}$	$b_f(2) = 2,$
$A_3 = \{3, 5, 8\}$	
$D_3 = \{1, 2, 3, 4, 5, 8\}$	$b_f(3) = 6,$
$A_4 = \{3, 5, 7, 8, 12, 15\}$	
$D_4 = \{1, 2, 3, 4, 5, 6, 7, 8, 12, 15\}$	$b_f(4) = 9,$
$A_5 = \{3, 5, 7, 8, 9, 12, 15, 16, 21, 24\}$	
$D_5 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 15, 16, 21, 24\}$	$b_f(5) = 10,$
$A_6 = \{3, 5, 7, 8, 9, 11, 12, 15, 16, 20, 21, 24, 25, 27, 32, 35\}$	
$D_6 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 16, 20, 21, 24, 27, 32, 35\}$	$b_f(6) = 13,$
$A_7 = \{3, 5, 7, 8, 9, 11, 12, 13, 15, 16, 20, 21, 24, 25, 27, 32, 33, 35, 45, 48\}$	
$D_7 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 20, 21, 24, 25, 27, 32, 33, 35, 45, 48\}$	$b_f(7) = 14,$
. . .	. . .

Note that $A_n = \{g(s) - g(r) : 1 \leq r < s \leq n\}$, where $g : \mathbb{N} \ni r \to r^2 \in \mathbb{N}$.

In this case $f(n_1, n_2) = (n_1 + n_2)^2 - n_1^2$ and $b_f(n)$ is equal to the discriminator $D_{r^2}(n)$.

Hence for $n > 4$ we get that $b_f(n)$ is the smallest $m \geq 2n$ such that m is a prime or twice a prime.

Definition 2.1

Let $f : \mathbb{N}^s \to \mathbb{N}$. We define for a natural number n.

$$b_f(n) = \tag{2}$$

$$\min\{m \in \mathbb{N} : \neg(\exists_{n_1, n_2, \ldots, n_s \in \mathbb{N}} \; m | f(n_1, n_2, \ldots, n_s) \wedge n_1 + n_2 + \ldots + n_s \leq n)\}.$$

or in other words:

For a given $n \geq 1$, D_n is the set of all divisors of all numbers $f(n_1, n_2, \ldots, n_s)$, where $n_1 + n_2 + \ldots + n_s \leq n$.
The numbers in D_n are cancelled, so the numbers in $\mathbb{N} \setminus D_n$ remain non-cancelled.
For $n \geq 1$ the least non canceled number we denote by $b_f(n)$.

Remark 2.2

Note that, for a natural number n, the set

$$\{m \in \mathbb{N} : \ \exists_{n_1, n_2, \ldots, n_s \in \mathbb{N}} \ m | f(n_1, n_2, \ldots, n_s) \ \wedge \ n_1 + n_2 + \ldots + n_s \leq n\}$$

is finite, thus

$$\{m \in \mathbb{N} : \ \neg(\exists_{n_1, n_2, \ldots, n_s \in \mathbb{N}} \ m | f(n_1, n_2, \ldots, n_s) \ \wedge \ n_1 + n_2 + \ldots + n_s \leq n)\}$$

is not empty. Therefore by minimum principle $b_f(n)$ exists.
We also have $b_f(1) = 1$ if $s > 1$.

the factorial function

Theorem 3.1

For the function $f : \mathbb{N} \ni t \to t! \in \mathbb{N}$ we get $b_f(3) = 4$ and

$$b_f(n) = \min\{m : m > n, \ m \text{ is a prime}\},$$

if $n \neq 3$.

[10] A. Tomski, M. Zakarczemny, *On some cancellation algorithms, III*, article in preparation.

Theorem 3.2

For the function $f : \mathbb{N} \ni t \to t(t+1) \in \mathbb{N}$ we have

$$b_f(n) = \min\{m : m > n+1, \ m = p^k, \ p \text{ is a prime}, \ k \in \mathbb{N}\}.$$

[9] A. Tomski, M. Zakarczemny, *On some cancellation algorithms*, NNTDM, 23 (2017), pp. 101-114.

Theorem 3.3

For the function $f : \mathbb{N} \ni t \to t(t+2) \in \mathbb{N}$ we get $b_f(1) = 2$ and

$$b_f(n) = \min\{m : m > n + 2, \ m = p^k \text{ or } m = 2p^k, \ p \text{ is an odd prime}, \ k \in \mathbb{N}\},$$

if $n > 1$.

[9] A. Tomski, M. Zakarczemny, *On some cancellation algorithms*, NNTDM, 23 (2017), pp. 101-114.

Theorem 3.4

We fix some integer $k \geq 2$. For the function $f : \mathbb{N} \ni t \to t^k \in \mathbb{N}$, we have

$$b_f(n) = \min\{m : m > n, \ m \text{ is a square} - \text{free}\}.$$

[9] A. Tomski, M. Zakarczemny, *On some cancellation algorithms*, NNTDM, 23 (2017), pp. 101-114.

Theorem 3.5

We fix some integer $k \geq 1$. For the function $f : \mathbb{N} \ni t \to tk^t \in \mathbb{N}$ we have that for $n > k$

$$b_f(n) = \min\{m : m > n, (k, m) = 1\}.$$

[10] A. Tomski, M. Zakarczemny, *On some cancellation algorithms, III*, article in preparation.

Theorem 3.6

For the function $f : \mathbb{N}^2 \ni (n_1, n_2, \ldots, n_s) \to n_1 n_2 \cdot \ldots \cdot n_s \in \mathbb{N}$,
where $s \geq 2$, we have $b_f(n) = 1$, if $s > n \geq 1$.
For $n \geq s$ we have

$$b_f(n) = \min\{m : m > n - s + 1, \ m \text{ is a prime}\}.$$

[10] A. Tomski, M. Zakarczemny, *On some cancellation algorithms, III*, article in preparation.

Theorem 3.7

For the function $f : \mathbb{N}^2 \ni (n_1, n_2) \to n_1^2 + n_2^2 \in \mathbb{N}$ *we have*

$$b_f(n) = \min\{m : 2m \geq n + 1, \ m \ square - free \ product \ of \ primes \equiv 3 \pmod 4\}.$$

[3] J. Browkin, H-Q. Cao, *Modifications of the Eratosthenes sieve*, Colloq. Math. 135, (2014), pp. 127-138.

Theorem 3.8

For the function $f : \mathbb{N}^3 \ni (n_1, n_2, n_3) \to n_1{}^2 + n_2{}^2 + n_3^2 \in \mathbb{N}$ we have

$$b_f(1) = 1, \ b_f(2) = 1, \ b_f(3) = 2, \ b_f(4) = 4, \ b_f(5) = 4.$$

Moreover, for any integer $s \geq 1$ we have:

1) *If $2 \cdot 2^s \leq n < 3 \cdot 2^s$, then $\frac{2\sqrt{3}}{3} \cdot 2^s < b_f(n) \leq 4^s$,*

2) *If $3 \cdot 2^s \leq n < 2 \cdot 2^{s+1}$, then $\sqrt{3} \cdot 2^s < b_f(n) \leq 5 \cdot 4^{s-1}$.*

Hurwitz theorem

The only natural numbers n for which n^2 is not the sum of the squares of three natural numbers are the numbers $n = 2^h$ and $n = 5 \cdot 2^h$, where $h = 0, 1, 2, \ldots$.
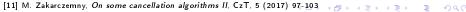
Theorem 3.9

For the function

$$f : \mathbb{N}^4 \ni (n_1, n_2, n_3, n_4) \to n_1{}^2 + n_2{}^2 + n_3{}^2 + n_4{}^2 \in \mathbb{N}$$

we have

$$b_f(1) = 1, \ b_f(2) = 1, \ b_f(3) = 1, \ b_f(4) = 3, \ b_f(5) = 3.$$

Moreover, for any integer $s \geq 1$ we have:

1) *If $3 \cdot 2^s \leq n < 4 \cdot 2^s$, then $b_f(n) \leq 2^{2s+1}$,*

2) *If $4 \cdot 2^s \leq n < 3 \cdot 2^{s+1}$, then $b_f(n) \leq 3 \cdot 2^{2s+1}$.*

[11] M. Zakarczemny, *On some cancellation algorithms II*, CzT, 5 (2017) 97-103

Theorem 3.10

For the function $f : \mathbb{N}^2 \ni (n_1, n_2) \to n_1{}^3 + n_2{}^3 \in \mathbb{N}$ we have

$$b_f(n) = \min\{m : m > n, \, m \, square - free, \, (3, \varphi(m)) = 1\}.$$

[9] A. Tomski, M. Zakarczemny, *On some cancellation algorithms*, NNTDM, 23 (2017), pp. 101-114.

Lemma 3.11

For a natural number $m > 4$ and an odd number $j \geq 3$ the following statements are equivalent

(i) For all $a, b \in \mathbb{N}$ such that $a + b \leq m - 1$ we have $m \nmid a^j + b^j$,

(ii) $(j, \varphi(m)) = 1$ and m is square-free,

(iii) x^j is a permutation polynomial of the finite ring $\mathbb{Z}/m\mathbb{Z}$.

It follows from [2,p.32] that (ii) and (iii) are equivalent.

Assume that (ii) holds. If there exist $a, b \in \mathbb{N}$ such that $a + b \leq m - 1$ and $a^j + b^j \equiv 0 \pmod{m}$, then $a^j \equiv (m - b)^j \pmod{m}$ and $1 \leq a < m - b \leq m - 1$. We obtain a contradiction with (iii). Hence (ii) implies (i).

On the other hand assume that (i) holds. Then for all $a, b \in \mathbb{N}$, $1 \leq a < b \leq m - 1$
we have following relations $m \nmid a^j + (m - b)^j$ and $m \nmid a^j - b^j$.
Hence $1^j, 2^j, \ldots, (m - 1)^j$ are distinct modulo m.
We will show that m is square-free. Suppose the contrary, we put
$m = p^2 l > 4$, where $l \in \mathbb{N}$ and p is a prime number. If we take

$$a = \left\{ \begin{array}{lll} pl - p & \text{if} & p = 2, l > 1 \\ pl & \text{if} & p \geq 3, l \geq 1 \end{array} \right. , \quad b = \left\{ \begin{array}{lll} p & \text{if} & p = 2, l > 1 \\ pl & \text{if} & p \geq 3, l \geq 1 \end{array} \right. ,$$

then $a + b \leq m - 1$ and $a^j + b^j \equiv 0 \pmod{m}$, thus we get contradiction with (i).
Consequently m is a square-free number.
Therefore $a^j \equiv 0 \pmod{m}$ implies $a \equiv 0 \pmod{m}$.

Thus $0^j, 1^j, \ldots, (m - 1)^j$ are distinct modulo m and (iii) holds, hence (ii) holds also.

[2] P. S. Bremser, P.D. Schumer, L.C. Washington, *A note on the incongruence of consecutive integers to a fixed power*, J. Number Theory (1990), 35, no. 1, 105-108.

Theorem 3.12

We fix some odd integer $j \geq 3$. For the function
$f : \mathbb{N}^2 \ni (n_1, n_2) \to n_1{}^j + n_2{}^j \in \mathbb{N}$ we have

$$n < b_f(n) \leq \min \{m : \ m > n, \ m \, square - free, \ (j, \varphi(m)) = 1\}.$$

Proof.

The first inequality follows from the fact that if j is an odd integer then $(n_1 + n_2) | n_1{}^j + n_2{}^j$.

Indeed, for a natural number $2 \leq h \leq n$, we take $n_1 = 1$, $n_2 = h - 1$. Hence $h | n_1{}^j + n_2{}^j$ and $n_1 + n_2 = h \leq n$. Therefore h is canceled. Hence $b_f(n) > n$.

For the proof of the second inequality assume that $m > n$, m is square-free number, $(j, \varphi(m)) = 1$, then by lemma 3.11 for all n_1, $n_2 \in \mathbb{N}$ such that $n_1 + n_2 \leq n \leq m - 1$ we have $m \nmid n_1^j + n_2^j$. Hence $b_f(n) \leq m$ and theorem follows. $\qquad\square$

Definition 3.13

For j odd, let B_j be the smallest integer such that for all $n > B_j$ there exists a prime p with $(j, p - 1) = 1$ and $p \in (n, \frac{3}{2}n)$.

Remark 3.14

By the Prime Number Theorem for arithmetic progressions there is always a prime $p \equiv 2 \pmod{j}$ in $(n, \frac{3}{2}n)$ for n sufficiently large, see [2].

[2] P. S. Bremser, P.D. Schumer, L.C. Washington, *A note on the incongruence of consecutive integers to a fixed power*, J. Number Theory (1990), 35, no. 1, 105-108.

Definition 3.15

For j odd, let Q_j be the set of all square-free positive integers m such that $(j, \varphi(m)) = 1$.

Lemma 3.16

For all $n > B_j$ there exists a number $q \in Q_j$ such that $q \in (n, \frac{3}{2}n)$.

Proof.

By definition of B_j and Q_j. $\qquad \square$

Theorem 3.17

We fix some odd integer $j \geq 3$. For the function
$f : \mathbb{N}^2 \ni (n_1, n_2) \to n_1{}^j + n_2{}^j \in \mathbb{N}$ *and for all $n > B_j$ we have*

$$n < b_f(n) < \tfrac{3}{2} n.$$

Moreover, if $n > \max\{3, B_j\}$ then $b_f(n)$ is a square-free number.

Proof.

By lemma 3.16, if $n > B_j$ we get

$$\min\left\{m : m > n, \, m \, square - free, \, (j, \varphi(m)) = 1\right\} < \tfrac{3}{2}n.$$

Thus by theorem 3.12 we obtain $n < b_f(n) < \tfrac{3}{2}n$.
By the straightforward verification for all odd $j \geq 3$ we have

$$b_f(1) = 1, \, b_f(2) = 3, \, b_f(3) = 4, \, b_f(4) = 5, \, b_f(5) = 6.$$

We will show that if $n > \max\{3, B_j\}$ then $b_f(n)$ is a square-free number.
Suppose the contrary $b_f(n) = p^2 l > 7$, where $l \in \mathbb{N}$ and p is a prime number. If $p = 2$ then $l \geq 2$ we put $a = 2l - 2$, $b = 2$ and get $b_f(n)|2^j((l-1)^j + 1)$, since j odd, $j \geq 3$. Hence $a + b = 2l = \tfrac{1}{2}b_f(n) < \tfrac{1}{2}(1 + \tfrac{1}{2})n < n$ and $b_f(n)|a^j + b^j$, thus we get contradiction with definition of $b_f(n)$.

If $p > 2$ then we put $a = pl$, $b = pl$ and get $b_f(n)|a^j + b^j$, since $j \geq 3$. But $a + b = 2pl = \tfrac{2}{p}b_f(n) < \tfrac{2}{p}(1 + \tfrac{1}{2})n \leq n$, thus we get contradiction with definition of $b_f(n)$. $\square$

Conjecture 3.18

We fix some odd integer $j \geq 3$. For the function
$f : \mathbb{N}^2 \ni (n_1, n_2) \rightarrow n_1{}^j + n_2{}^j \in \mathbb{N}$, if a natural number $n \geq 4$ then

$$b_f(n) = \min\{m : \ m > n, \ m \ square - free, \ (j, \varphi(m)) = 1\} \tag{3}$$

$$= \min\{m : \ m > n, \ polynomial \ x^j \ permutes \ elements \ of \ \mathbb{Z}/m\mathbb{Z}\}.$$

Remark 3.19

For proof of Conjecture 3.18 in the case $j = 3$, see Theorem 3.10.
The author found that the equation (3) holds for $j \in \{5, 7, 9, 11, 13\}$ and
$n \in \{4, 5, \ldots, 200\}$.

Consider an arbitrary function $f : \mathbb{N}^m \to \mathbb{N}$ and the set

$$V_n = \{f(n_1, n_2, \ldots, n_m) : \ n_1 + n_2 + \ldots + n_m \leq n\}.$$

Cancel in $\mathbb{N}$ all numbers $d \in \mathbb{N}$ such that d^2 is a divisor of some number in V_n and define $b_f^{(2)}(n)$ as the least non-canceled number.

$f(n_1, n_2) = n_1{}^2 + n_2{}^2$ and $b_f^{(2)}$

Denote by F the set of all positive integers which are the products of prime numbers $\not\equiv 1 \pmod 4$.
Let $(q_s)_{s=1}^{\infty}$ be the increasing sequence of all elements of F.
In particular, $q_1 = 1$, which corresponds to the empty product.

$$F = \{1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 14, 16, 18, 19, 21, 22, 23, 24, 27, 28, 31, \ldots\}.$$

Theorem 4.1

Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $f(n_1, n_2) = n_1{}^2 + n_2{}^2$. We have $b_f^{(2)}(1) = 1$ and for $n \geq 2$

$$b_f^{(2)}(n) = q_s, \text{ if } 2q_{s-1} \leq n < 2q_s,$$

where $s \geq 2$.
Hence, the set $\{b_f^{(2)}(n) : n \in \mathbb{N}\}$ is equal to F.

[9] A. Tomski, M. Zakarczemny, *On some cancellation algorithms*, NNTDM, 23 (2017), pp. 101-114.

$f(n_1, n_2, n_3) = n_1{}^2 + n_2{}^2 + n_3{}^2$ and $b_f^{(2)}$

Theorem 4.2

For the function $f : \mathbb{N}^3 \to \mathbb{N}$ given by the formula
$f(n_1, n_2, n_3) = n_1{}^2 + n_2{}^2 + n_3{}^2$, we have $b_f^{(2)}(1) = 1$, $b_f^{(2)}(2) = 1$, and for $n \geq 3$

$$b_f^{(2)}(n) \leq 2^{\left\lceil \log_2 \frac{n}{3} \right\rceil}.$$

Remark 4.3

We conjecture that for any $n \geq 3$ we have $b_f^{(2)}(n) = 2^{\left\lceil \log_2 \frac{n}{3} \right\rceil}$.

[9] A. Tomski, M. Zakarczemny, *On some cancellation algorithms*, NNTDM, 23 (2017), pp. 101-114.

$$f(n_1, n_2, n_3) = n_1{}^3 + n_2{}^3 + n_3{}^3$$

Problem 5.1

For the function $f : \mathbb{N}^3 \to \mathbb{N}$ given by the formula
$f(n_1, n_2, n_3) = n_1{}^3 + n_2{}^3 + n_3{}^3$. *We have*

n	1, 2	3	4, 5	6, ..., 10	11, ..., 17	18, 19	20, ..., 24	25, 26	27, 28, 29	30, ..., 34
$b_f(n)$	1	2	4	7	13	52	65	117	156	169

n	35, 36, 37	38, ..., 41	42, ..., 48	49, ..., 57	58, 59	60, 61, 62	63, ..., 66	67, ..., 73
$b_f(n)$	241	260	301	481	802	903	973	1118

Find and prove an explicit formula for the above sequence.

$$f(n_1, n_2, n_3, n_4, n_5) = n_1{}^2 + n_2{}^2 + n_3{}^2 + n_4{}^2 + n_5{}^2$$

Problem 5.2

For the function $f : \mathbb{N}^5 \to \mathbb{N}$ given by the formula
$f(n_1, n_2, n_3, n_4, n_5) = n_1{}^2 + n_2{}^2 + n_3{}^2 + n_4{}^2 + n_5{}^2$. We have

n	1, 2, 3, 4	5	6, 7, 8	9	10	11	12, 13, 14, 15	16	17	18, 19, 20	21	22	23, 24
$b_f(n)$	1	2	3	6	9	15	33	73	90	105	132	153	193

n	25	26	27	28	29	30	31, 32	33	34	35, 36	37	38, 39, 40	41	42
$b_f(n)$	210	225	288	297	318	321	353	432	441	513	570	585	732	793

n	43, 44, 45, 46	47, 48	49, 50	51	52	53, 54	55, 56	57	58	59, 60	61	
$b_f(n)$	825	1065	1185	1212	1257	1425	1473	1500	1617	1737	1860	

Find and prove an explicit formula for the above sequence.

The only natural numbers that are not the sums of the squares of five natural numbers are the numbers
1, 2, 3, 4, 6, 7, 9, 10, 12, 15, 18, 33.

$f(n_1, n_2, n_3) = \frac{n_1(n_1+1)}{2} + \frac{n_2(n_2+1)}{2} + \frac{n_3(n_3+1)}{2}$, sum of three triangular numbers

Problem 5.3

For the function $f : \mathbb{N}^3 \to \mathbb{N}$ given by the formula
$f(n_1, n_2, n_3) = \frac{n_1(n_1+1)}{2} + \frac{n_2(n_2+1)}{2} + \frac{n_3(n_3+1)}{2}$. *We have*

n	1, 2	3, 4	5	6, 7, 8	9, 10	11, 12, 13, 14	15	16	17	18, 19
$b_f(n)$	1	2	6	11	20	29	53	69	76	81

n	20	21	22	23, 24	25	26, 27	28	29, 30	31, 32, 33	34
$b_f(n)$	105	106	110	119	146	179	188	218	254	272

Find and prove an explicit formula for the above sequence.

Gauss was the first to prove that every natural number which is not of the form $4^l(8k + 7)$, k and l being non-negative integers, is the sum of the squares of three integers.
The theorem of Gauss implies a theorem (first formulated by Fermat) stating that any natural number is the sum of three or fewer triangular numbers.

$f(t) = F_t$, the Fibonacci numbers

Problem 5.4

For the function $f : \mathbb{N} \to \mathbb{N}$ given by the formula $f(t) = F_t$. We have

n	1, 2	3	4, 5	6, ..., 11	12, 13, 14	15, ..., 23	24, ..., 29	30, ..., 35	36, ..., 43	44
$b_f(n)$	2	3	4	6	10	14	20	27	30	43

we recall that, in this case:

For a given $n \geq 1$, D_n is the set of all divisors of all numbers $F_1, F_2, F_3, \ldots, F_n$.
For $n \geq 1$ the least number in the set $\mathbb{N} \setminus D_n$ we denote by $b_f(n)$.

For all $n \geq 1$ we have that $n | F_1 F_2 \cdot \ldots \cdot F_{n^2}$, hence $b_f(n^2) \geq n$.

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Thank you for your attention.