

Number of solutions in a box of a linear equation in an Abelian group

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Karol Cwalina and Tomasz Schoen [1] have recently proved the following conjecture of Andrzej Schinzel [4]: the number of solutions of the congruence

$$a_1x_1 + \dots + a_kx_k \equiv 0 \pmod{n}$$

in the box $0 \leq x_i \leq b_i$, where b_i are positive integers, is at least

$$2^{1-n} \prod_{i=1}^k (b_i + 1).$$

Using a completely different method we shall prove the following more general statement, also conjectured by Schinzel [4].

Theorem 1.1.

For every finite Abelian group Γ , for all $a_1, \dots, a_k \in \Gamma$, and for all positive integers b_1, \dots, b_k the number of solutions of the equation

$$\sum_{i=1}^k a_i x_i = 0$$

in non-negative integers $x_i \leq b_i$ is at least

$$2^{1-D(\Gamma)} \prod_{i=1}^k (b_i + 1), \quad (1)$$

where $D(\Gamma)$ is the Davenport constant of the group Γ (see Definition 2.1. below).

Let Γ be a finite Abelian group, with multiplicative notation.

Definition 2.1.

Define the *Davenport constant* $D(\Gamma)$ to be the smallest positive integer n such that, for any sequence g_1, \dots, g_n of group elements, there exist a non-empty sequence of indices

$$1 \leq i_1 < \dots < i_t \leq n$$

such that

$$g_{i_1} \cdot \dots \cdot g_{i_t} = 1.$$

For a group with multiplicative notation, Theorem 1.1 has the form:
 for every finite Abelian group Γ , for all $a_1, \dots, a_k \in \Gamma$, and for all positive integers b_1, \dots, b_k the number of solutions of the equation

$$\prod_{i=1}^k a_i^{x_i} = 1$$

in non-negative integers $x_i \leq b_i$ is at least

$$2^{1-D(\Gamma)} \prod_{i=1}^k (b_i + 1). \quad (2)$$

By the definition of the Davenport constant, we may find $g_1, \dots, g_{D(\Gamma)-1} \in \Gamma$ such that any product of a non-empty subsequence of this sequence is not equal 1 in Γ .

Since the number of solutions of the equation $\prod_{i=1}^{D(\Gamma)-1} g_i^{x_i} = 1$, where $x_i = 0$ or

$x_i = 1$, is equal $1 = 2^{1-D(\Gamma)} \prod_{i=1}^{D(\Gamma)-1} (1 + 1)$ we obtain:

Remark 2.2.

In Theorem 1.1, $2^{1-D(\Gamma)}$ is the best possible coefficient independent of a_i , b_i and dependent only on Γ .

Lemma 2.3.

For $n \geq 1$ we have the following identity in $\mathbb{Q}[x]$ and in the group ring $\mathbb{Q}[\Gamma]$.

$$1 + x + x^2 + \dots + x^n = \sum_{j=0}^n 2^{j-n-1} (1 + x^j) (1 + x)^{n-j}. \quad (3)$$

Proof. We proceed by induction on n .

(Elements of $\mathbb{Q}[\Gamma]$ are sometimes written as what are called "formal linear combinations of elements of Γ , with coefficients in \mathbb{Q} " where this doesn't cause confusion)

Definition 2.4.

For an element $\sum_{g \in \Gamma} N_g g$ of the group ring $\mathbb{Q}[\Gamma]$ and a number $n \in \mathbb{Q}$ we write

$$\sum_{g \in \Gamma} N_g g \succeq n \text{ iff } N_1 \geq n.$$

Lemma 2.5.

Theorem 1.1 in multiplicative notation is equivalent to the statement:
for every finite Abelian group Γ , for all $a_1, \dots, a_k \in \Gamma$, and for all positive integers b_1, \dots, b_k we have relation:

$$\prod_{i=1}^k (1 + a_i + \dots + a_i^{b_i}) \succeq 2^{1-D(\Gamma)} \prod_{i=1}^k (b_i + 1), \quad (4)$$

where $D(\Gamma)$ is the Davenport constant of the group Γ .

Proof. Indeed, the number of solutions of the equation $\prod_{i=1}^k a_i^{x_i} = 1$ in non-negative integers $x_i \leq b_i$ is equal to N_1 , where

$$\prod_{i=1}^k (1 + a_i + \dots + a_i^{b_i}) = \sum_{g \in \Gamma} N_g g.$$

We have $N_1 \geq 2^{1-D(\Gamma)} \prod_{i=1}^k (b_i + 1)$ if and only if relation (4) holds.

Lemma 2.6.

Let Γ be a finite Abelian group. For all $a_1, \dots, a_k \in \Gamma$ we have

$$(1 + a_1)(1 + a_2) \cdot \dots \cdot (1 + a_k) \succeq 2^{1-D(\Gamma)} \cdot 2^k. \quad (5)$$

Proof. For the completeness of the exposition we provide Olson's proof [3]. We proceed by induction on k . For $k \leq D(\Gamma) - 1$ we have

$$(1 + a_1)(1 + a_2) \cdot \dots \cdot (1 + a_k) \succeq 1 \geq 2^{1-D(\Gamma)} \cdot 2^k$$

and the assertion is true.

Assume it is true for the number of factors less than k , where $k > D(\Gamma) - 1$. Hence $k \geq D(\Gamma)$. By the definition of the Davenport constant we may assume, without loss of generality, that

$$a_1 \cdot \dots \cdot a_t = 1, \text{ for some } 1 \leq t \leq D(\Gamma).$$

By the inductive assumption

$$\prod_{i=2}^t (1 + a_i^{-1}) \prod_{i=t+1}^k (1 + a_i) \succeq 2^{1-D(\Gamma)} \cdot 2^{k-1},$$

$$\prod_{i=2}^k (1 + a_i) \succeq 2^{1-D(\Gamma)} \cdot 2^{k-1}.$$

Hence

$$\begin{aligned}
 \prod_{i=1}^k (1 + a_i) &= \prod_{i=2}^k (1 + a_i) + a_1 \prod_{i=2}^k (1 + a_i) \\
 &= \prod_{i=2}^k (1 + a_i) + a_1 a_2 \cdot \dots \cdot a_t \prod_{i=2}^t (1 + a_i^{-1}) \prod_{i=t+1}^k (1 + a_i) \\
 &= \prod_{i=2}^k (1 + a_i) + \prod_{i=2}^t (1 + a_i^{-1}) \prod_{i=t+1}^k (1 + a_i) \succeq 2^{1-D(\Gamma)} \cdot 2^{k-1} + 2^{1-D(\Gamma)} \cdot 2^{k-1} = 2^{1-D(\Gamma)} \cdot 2^k.
 \end{aligned}$$

By Lemma 2.5. it suffices to prove:

Theorem

For every finite Abelian group Γ , for all $a_1, \dots, a_k \in \Gamma$, and for all positive integers b_1, \dots, b_k we have

$$\prod_{i=1}^k (1 + a_i + \dots + a_i^{b_i}) \succeq 2^{1-D(\Gamma)} \prod_{i=1}^k (b_i + 1),$$

where $D(\Gamma)$ is the Davenport constant of the group Γ .

Proof. We use the identity (3) to get

$$\begin{aligned}
 \prod_{i=1}^k (1 + a_i + \dots + a_i^{b_i}) &= \prod_{i=1}^k \sum_{j=0}^{b_i} 2^{j-b_i-1} (1 + a_i^j) (1 + a_i)^{b_i-j} \\
 &= \sum_{\substack{0 \leq j_1 \leq b_1 \\ 0 \leq j_2 \leq b_2 \\ \vdots \\ 0 \leq j_k \leq b_k}} \prod_{i=1}^k 2^{j_i-b_i-1} (1 + a_i^{j_i}) (1 + a_i)^{b_i-j_i}.
 \end{aligned} \tag{6}$$

By Lemma 2.6. we obtain

$$\begin{aligned}
 & \sum_{\substack{0 \leq j_1 \leq b_1 \\ 0 \leq j_2 \leq b_2 \\ \vdots \\ 0 \leq j_k \leq b_k}} \prod_{i=1}^k 2^{j_i - b_i - 1} (1 + a_i^{j_i}) (1 + a_i)^{b_i - j_i} \\
 & \succeq 2^{1-D(\Gamma)} \sum_{\substack{0 \leq j_1 \leq b_1 \\ 0 \leq j_2 \leq b_2 \\ \vdots \\ 0 \leq j_k \leq b_k}} \prod_{i=1}^k 2^{j_i - b_i - 1} 2^{1+b_i-j_i} = 2^{1-D(\Gamma)} \sum_{\substack{0 \leq j_1 \leq b_1 \\ 0 \leq j_2 \leq b_2 \\ \vdots \\ 0 \leq j_k \leq b_k}} 1 \\
 & = 2^{1-D(\Gamma)} \prod_{i=1}^k (b_i + 1).
 \end{aligned}$$

Thus

$$\prod_{i=1}^k (1 + a_i + \dots + a_i^{b_i}) \succeq 2^{1-D(\Gamma)} \prod_{i=1}^k (b_i + 1).$$

We have proved in [9] the following two statements.

Theorem 3.1.

For every finite Abelian group Γ , for all $g, a_1, \dots, a_k \in \Gamma$, if there exists a solution of the equation $\sum_{i=1}^k a_i x_i = g$ in non-negative integers $x_i \leq b_i$, where b_i are positive integers, then the number of such solutions is at least

$$3^{1-D(\Gamma)} \prod_{i=1}^k (b_i + 1). \quad (7)$$

Remark 3.2.

Let $\Gamma = n\mathbb{Z}_2$ be a direct product of n cyclic groups of order two, a_1, \dots, a_n a basis for Γ . Then the number of solutions of the equation

$$\sum_{i=1}^n a_i x_i = \sum_{i=1}^n a_i$$

in non-negative integers $x_i \leq b_i = 2$, equals 1.

Since $D(\Gamma) = n + 1$ (see Olson [2]) and $1 = 3^{1-D(\Gamma)} \prod_{i=1}^n (2 + 1)$, we get that $3^{1-D(\Gamma)}$ is the best possible coefficient independent of a_i, b_i, g and dependent only on Γ .

Theorem 3.3.

For every finite Abelian group Γ , for all $g, a_1, \dots, a_k \in \Gamma$, if there exists a solution of the equation $\sum_{i=1}^k a_i x_i = g$ in non-negative integers $x_i \leq b_i$, where $b_i \in \{2^s - 1 : s \in \mathbb{N}\}$, then the number of such solutions is at least

$$2^{1-D(\Gamma)} \prod_{i=1}^k (b_i + 1). \quad (8)$$

Lemma 3.4.

For every finite Abelian group Γ with multiplicative notation and for all $a_1, \dots, a_k, g \in \Gamma$, the number of solutions of the equation $\prod_{i=1}^k a_i^{x_i} = g$ in non-negative integers $x_i \leq b_i$ is equal to N_1 , where

$$g^{-1} \prod_{i=1}^k (1 + a_i + \dots + a_i^{b_i}) = \sum_{h \in \Gamma} N_h h,$$

is an identity in $\mathbb{Q}[\Gamma]$.

Proof. We interpret the equation $g^{-1} \prod_{i=1}^k (1 + a_i + \dots + a_i^{b_i}) = \sum_{h \in \Gamma} N_h h$ combinatorially. For $g \in \Gamma$ look at all sequences $a_1^{x_1}, a_2^{x_2}, \dots, a_k^{x_k}$, that have product g , where $x_i \leq b_i$ are non-negative integers. Then N_1 count those sequences. Therefore the number of solutions of the equation $\prod_{i=1}^k a_i^{x_i} = g$ in non-negative integers $x_i \leq b_i$ is equal to N_1 .

Lemma 3.5.

Theorem 3.1. with multiplicative notation is equivalent to the statement: for every finite Abelian group Γ , for all $g, a_1, \dots, a_k \in \Gamma$, if there exists a solution of the equation $\prod_{i=1}^k a_i^{x_i} = g$ in non-negative integers $x_i \leq b_i$, where b_i are positive integers, then we have:

$$g^{-1} \prod_{i=1}^k (1 + a_i + \dots + a_i^{b_i}) \succeq 3^{1-D(\Gamma)} \prod_{i=1}^k (b_i + 1), \quad (9)$$

where $D(\Gamma)$ is the Davenport constant of the group Γ .

Proof. This follows from Lemma 3.4 and Definition 2.4.

Lemma 3.6.

Theorem 3.3. with multiplicative notation is equivalent to the statement: for every finite Abelian group Γ , for all $g, a_1, \dots, a_k \in \Gamma$, and for all positive integers $b_1, b_2, \dots, b_k \in \{2^s - 1 : s \in \mathbb{N}\}$, if there exists a solution of the equation

$\prod_{i=1}^k a_i^{x_i} = g$ in non-negative integers $x_i \leq b_i$, then we have relation:

$$g^{-1} \prod_{i=1}^k (1 + a_i + \dots + a_i^{b_i}) \succeq 2^{1-D(\Gamma)} \prod_{i=1}^k (b_i + 1). \quad (10)$$

Proof. This follows from Lemma 3.4 and Definition 2.4.

Lemma 3.7.

For every finite Abelian group Γ and for all $g, a_1, a_2, \dots, a_k \in \Gamma$, if there exists a solution of the equation $\prod_{i=1}^k a_i^{x_i} = g$ in non-negative integers $x_i \leq 1$, then

$$g^{-1} \prod_{i=1}^k (1 + a_i) \succeq 2^{1-D(\Gamma)} \cdot 2^k. \quad (11)$$

Proof. We may assume that $\prod_{i=1}^t a_i = g$, where $1 \leq t \leq k$.

We have the identities

$$g^{-1} \prod_{i=1}^k (1 + a_i) = g^{-1} \prod_{i=1}^t a_i \prod_{i=1}^t (1 + a_i^{-1}) \prod_{i=t+1}^k (1 + a_i) = \prod_{i=1}^t (1 + a_i^{-1}) \prod_{i=t+1}^k (1 + a_i).$$

By Theorem 1.1

$$\prod_{i=1}^t (1 + a_i^{-1}) \prod_{i=t+1}^k (1 + a_i) \succeq 2^{1-D(\Gamma)} 2^k.$$

This implies

$$g^{-1} \prod_{i=1}^k (1 + a_i) \succeq 2^{1-D(\Gamma)} 2^k.$$

Lemma 3.8.

If $0 \leq t < b$, where t, b are integers, then $b - t + 1 \geq (\frac{2}{3})^t(b + 1)$.

Proof. We verify by differentiation that the function $f(x) = 2(\frac{3}{2})^x - x - 2$ is increasing in the interval $(1, \infty)$. Since $f(0) = f(1) = 0$, $f(2) = \frac{1}{2}$ we get $2(\frac{3}{2})^t \geq t + 2$ for non-negative integers t . Hence $1 - \frac{t}{b+1} \geq 1 - \frac{t}{t+2} \geq (\frac{2}{3})^t$, and thus $b - t + 1 \geq (\frac{2}{3})^t(b + 1)$.

Lemma 3.9.

For $s \geq 1$ we have the following identity in $\mathbb{Q}[\Gamma]$:

$$1 + x + x^2 + \dots + x^{2^s - 1} = \prod_{j=1}^s (1 + x^{2^{j-1}}). \quad (12)$$

Proof. We proceed by induction on s .

Proof of Theorem 3.1.

We may find $0 \leq t_i \leq b_i$, where $1 \leq i \leq k$, such that $a_1^{t_1} a_2^{t_2} \cdot \dots \cdot a_k^{t_k} = g$.
By definition of the Davenport constant we may assume that

$$\sum_{i=1}^k t_i \leq D(\Gamma) - 1. \quad (13)$$

Let $t_i = b_i$ for $1 \leq i \leq s \leq k$; $t_i < b_i$ for $s+1 \leq i \leq k$;
if $t_i < b_i$ for $1 \leq i \leq k$, then we take $s = 0$.

We have the identities

$$\begin{aligned}
 & g^{-1} \prod_{i=1}^s (1 + a_i + \dots + a_i^{b_i}) \prod_{i=s+1}^k (a_i^{t_i} + a_i^{t_i+1} + \dots + a_i^{b_i}) = \\
 & = \left(\left(\prod_{i=1}^s a_i^{b_i} \right) \left(\prod_{i=s+1}^k a_i^{t_i} \right) \right)^{-1} \prod_{i=1}^s (1 + a_i + \dots + a_i^{b_i}) \prod_{i=s+1}^k (a_i^{t_i} + a_i^{t_i+1} + \dots + a_i^{b_i}) = \\
 & = \prod_{i=1}^s (1 + a_i^{-1} + \dots + (a_i^{-1})^{b_i}) \prod_{i=s+1}^k (1 + a_i + \dots + a_i^{b_i - t_i}).
 \end{aligned}$$

By Theorem 1.1.

$$\prod_{i=1}^s (1 + a_i^{-1} + \dots + (a_i^{-1})^{b_i}) \prod_{i=s+1}^k (1 + a_i + \dots + a_i^{b_i - t_i})$$

$$\succeq 2^{1-D(\Gamma)} \left(\prod_{i=1}^s (b_i + 1) \right) \left(\prod_{i=s+1}^k (b_i - t_i + 1) \right).$$

We have by Lemma 3.8. that

$$\begin{aligned}
 & 2^{1-D(\Gamma)} \left(\prod_{i=1}^s (b_i + 1) \right) \left(\prod_{i=s+1}^k (b_i - t_i + 1) \right) \\
 & \geq 2^{1-D(\Gamma)} \left(\prod_{i=1}^s (b_i + 1) \right) \left(\prod_{i=s+1}^k \left(\frac{2}{3} \right)^{t_i} (b_i + 1) \right) = \\
 & = 2^{1-D(\Gamma)} \left(\frac{2}{3} \right)^{\sum_{i=s+1}^k t_i} \prod_{i=1}^k (b_i + 1) \geq 2^{1-D(\Gamma)} \left(\frac{2}{3} \right)^{\sum_{i=1}^k t_i} \prod_{i=1}^k (b_i + 1).
 \end{aligned}$$

Since (13) it follows that

$$2^{1-D(\Gamma)} \left(\frac{2}{3}\right)^{\sum_{i=1}^k t_i} \prod_{i=1}^k (b_i + 1) \geq 2^{1-D(\Gamma)} \left(\frac{2}{3}\right)^{D(\Gamma)-1} \prod_{i=1}^k (b_i + 1) = 3^{1-D(\Gamma)} \prod_{i=1}^k (b_i + 1).$$

Hence

$$g^{-1} \prod_{i=1}^s (1 + a_i + \dots + a_i^{b_i}) \prod_{i=s+1}^k (a_i^{t_i} + a_i^{t_i+1} + \dots + a_i^{b_i}) \succeq 3^{1-D(\Gamma)} \prod_{i=1}^k (b_i + 1).$$

Finally

$$g^{-1} \prod_{i=1}^k (1 + a_i + \dots + a_i^{b_i}) \succeq 3^{1-D(\Gamma)} \prod_{i=1}^k (b_i + 1).$$

Proof of Theorem 3.3.

Let $b_i = 2^{s_i} - 1$, where $s_i \in \mathbb{N}$.

We take $0 \leq t_i \leq b_i$, where $1 \leq i \leq k$ such that $a_1^{t_1} a_2^{t_2} \cdot \dots \cdot a_k^{t_k} = g$.

Since $0 \leq t_i \leq 2^{s_i} - 1$ we may find $\epsilon_{ji} \in \{0, 1\}$ such that

$$t_i = \sum_{j=1}^{s_i} \epsilon_{ji} 2^{j-1}$$

for $1 \leq i \leq k$.

Using (12) we obtain

$$\begin{aligned}
 a_i^{-t_i}(1 + a_i + \dots + a_i^{b_i}) &= a_i^{-t_i} \prod_{j=1}^{s_i} (1 + a_i^{2^{j-1}}) = \\
 &= a_i^{-\sum_{j=1}^{s_i} \epsilon_{ji} 2^{j-1}} \prod_{j=1}^{s_i} (1 + a_i^{2^{j-1}}) = \prod_{j=1}^{s_i} a_i^{-\epsilon_{ji} 2^{j-1}} \prod_{j=1}^{s_i} (1 + a_i^{2^{j-1}}) = \prod_{j=1}^{s_i} a_i^{-\epsilon_{ji} 2^{j-1}} (1 + a_i^{2^{j-1}}) = \\
 &= \prod_{j=1}^{s_i} (a_i^{-\epsilon_{ji} 2^{j-1}} + a_i^{(1-\epsilon_{ji}) 2^{j-1}}) = \prod_{j=1}^{s_i} (1 + a_i^{\eta_{ji} 2^{j-1}}),
 \end{aligned}$$

where $\eta_{ji} = 1 - 2\epsilon_{ji} \in \{-1, 1\}$.

Thus

$$g^{-1} \prod_{i=1}^k (1 + a_i + \dots + a_i^{b_i}) = \prod_{i=1}^k a_i^{-t_i} (1 + a_i + \dots + a_i^{b_i}) = \prod_{i=1}^k \prod_{j=1}^{s_i} (1 + a_i^{\eta_{ji} 2^{j-1}}).$$

By Theorem 1.1.

$$\prod_{i=1}^k \prod_{j=1}^{s_i} (1 + a_i^{\eta_{ji} 2^{j-1}}) \succeq 2^{1-D(\Gamma)} \prod_{i=1}^k \prod_{j=1}^{s_i} 2 = 2^{1-D(\Gamma)} \prod_{i=1}^k 2^{s_i} = 2^{1-D(\Gamma)} \prod_{i=1}^k (b_i + 1),$$

which implies

$$g^{-1} \prod_{i=1}^k (1 + a_i + \dots + a_i^{b_i}) \succeq 2^{1-D(\Gamma)} \prod_{i=1}^k (b_i + 1).$$

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Thank you for your attention.

Theorem 1.1 we may rewrite in the form:

for all positive integers $n_1 \mid n_2 \mid \dots \mid n_l$, b_i and for all integers a_{ij} , where $1 \leq i \leq k$, $1 \leq j \leq l$ the number of solutions of the system

$$\left\{ \begin{array}{l} a_{11}x_1 + a_{21}x_2 + \dots + a_{k1}x_k \equiv 0 \pmod{n_1}, \\ a_{12}x_1 + a_{22}x_2 + \dots + a_{k2}x_k \equiv 0 \pmod{n_2}, \\ \vdots \\ a_{1l}x_1 + a_{2l}x_2 + \dots + a_{kl}x_k \equiv 0 \pmod{n_l}, \end{array} \right.$$

in non-negative integers $x_i \leq b_i$ is at least

$$2^{1-D(\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \dots \oplus \mathbb{Z}_{n_l})} \prod_{i=1}^k (b_i + 1).$$

Group ring

Group ring $\mathbb{Q}[\Gamma]$ is a \mathbb{Q} -vector space with basis Γ and with multiplication defined distributively using the given multiplication of Γ .

$$\left(\sum_{g \in \Gamma} \alpha_g g \right) \cdot \left(\sum_{g \in \Gamma} \beta_g g \right) = \sum_{x \in \Gamma} \left(\sum_{gh=x} \alpha_g \beta_h \right) x.$$

We have $\sum_{g \in \Gamma} \alpha_g g = \sum_{g \in \Gamma} \beta_g g$ iff $\alpha_g = \beta_g$ for all $g \in \Gamma$.

Instead $\sum_{g \in \Gamma} 0g$ we write 0.

Instead $1g$ we write g .

Instead $(-\alpha)g$ we write $-\alpha g$.

We denote the group unit 1_Γ and the unit element of the ring \mathbb{Q} by the same symbol 1.

We denote addition operation in $\mathbb{Q}[\Gamma]$ and in \mathbb{Q} by the same symbol.

If $1_\Gamma = 1$, then the additive group of $\mathbb{Q}[\Gamma]$ becomes an extension of the additive group of \mathbb{Q} , thus the use of the same symbol $+$ is legitimate.