Number of solutions in a box of a linear equation in an Abelian group

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Karol Cwalina and Tomasz Schoen [1] have recently proved the following conjecture of Andrzej Schinzel [4]: the number of solutions of the congruence

$$a_1x_1 + \ldots + a_kx_k \equiv 0 \,(\bmod n)$$

in the box $0 \leq x_i \leq b_i$, where b_i are positive integers, is at least

$$2^{1-n}\prod_{i=1}^{k}(b_i+1).$$

Using a completely different method we shall prove the following more general statement, also conjectured by Schinzel [4].

Theorem 1.1.

For every finite Abelian group Γ, for all $a_1, \ldots, a_k \in \Gamma$, and for all positive integers $b_1, \ldots, b_k$ the number of solutions of the equation

$$\sum_{i=1}^{k} a_i x_i = 0$$

in non-negative integers $x_i \leq b_i$ is at least

$$2^{1-D(\Gamma)} \prod_{i=1}^{k} (b_i + 1), \tag{1}$$

where $D(\Gamma)$ is the Davenport constant of the group Γ (see Definition 2.1. below).

Let Γ be a finite Abelian group, with multiplicative notation.

Defnition 2.1.

Define the *Davenport constant* $D(\Gamma)$ to be the smallest positive integer n such that, for any sequence $g_1, \ldots, g_n$ of group elements, there exist a non-empty sequence of indices

$$1 \leq i_1 < \ldots < i_t \leq n$$

such that

$$g_{i_1} \cdot \ldots \cdot g_{i_t} = 1.$$

For a group with multiplicative notation, Theorem 1.1 has the form:
for every finite Abelian group Γ, for all $a_1, \ldots, a_k \in \Gamma$, and for all positive integers $b_1, \ldots, b_k$ the number of solutions of the equation

$$\prod_{i=1}^{k} a_i^{x_i} = 1$$

in non-negative integers $x_i \leq b_i$ is at least

$$2^{1-D(\Gamma)} \prod_{i=1}^{k} (b_i + 1). \tag{2}$$

By the definition of the Davenport constant, we may find $g_1, \ldots, g_{D(\Gamma)-1} \in \Gamma$ such that any product of a non-empty subsequence of this sequence is not equal 1 in Γ.

Since the number of solutions of the equation $\prod_{i=1}^{D(\Gamma)-1} g_i^{x_i} = 1$, where $x_i = 0$ or $x_i = 1$, is equal $1 = 2^{1-D(\Gamma)} \prod_{i=1}^{D(\Gamma)-1} (1+1)$ we obtain:

Remark 2.2.

In Theorem 1.1, $2^{1-D(\Gamma)}$ is the best possible coefficient independent of a_i, b_i and dependent only on Γ.

Lemma 2.3.

For $n \geq 1$ we have the following identity in $\mathbb{Q}[x]$ and in the group ring $\mathbb{Q}[\Gamma]$.

$$1 + x + x^2 + \ldots + x^n = \sum_{j=0}^{n} 2^{j-n-1}(1 + x^j)(1 + x)^{n-j}. \tag{3}$$

Proof. We proceed by induction on n.

(Elements of $\mathbb{Q}[\Gamma]$ are sometimes written as what are called "formal linear combinations of elements of Γ, with coefficients in $\mathbb{Q}$" where this doesn't cause confusion)

Definition 2.4.

For an element $\sum_{g\in\Gamma} N_g g$ of the group ring $\mathbb{Q}[\Gamma]$ and a number $n \in \mathbb{Q}$ we write

$$\sum_{g\in\Gamma} N_g g \succeq n \ \text{ iff } \ N_1 \geq n.$$

Lemma 2.5.

Theorem 1.1 in multiplicative notation is equivalent to the statement:
for every finite Abelian group Γ, for all $a_1, \ldots, a_k \in \Gamma$, and for all positive integers $b_1, \ldots, b_k$ we have relation:

$$\prod_{i=1}^{k}(1 + a_i + \ldots + {a_i}^{b_i}) \succeq 2^{1-D(\Gamma)} \prod_{i=1}^{k}(b_i + 1), \tag{4}$$

where $D(\Gamma)$ is the Davenport constant of the group Γ.

Proof. Indeed, the number of solutions of the equation $\prod_{i=1}^{k} a_i^{x_i} = 1$ in non-negative integers $x_i \leq b_i$ is equal to N_1, where

$$\prod_{i=1}^{k}(1 + a_i + \ldots + {a_i}^{b_i}) = \sum_{g \in \Gamma} N_g g.$$

We have $N_1 \geq 2^{1-D(\Gamma)} \prod_{i=1}^{k}(b_i + 1)$ if and only if relation (4) holds.

Lemma 2.6.

Let Γ be a finite Abelian group. For all $a_1, \ldots, a_k \in \Gamma$ we have

$$(1 + a_1)(1 + a_2) \cdot \ldots \cdot (1 + a_k) \succeq 2^{1-D(\Gamma)} \cdot 2^k. \tag{5}$$

Proof. For the completeness of the exposition we provide Olson's proof [3]. We proceed by induction on k. For $k \leq D(\Gamma) - 1$ we have

$$(1 + a_1)(1 + a_2) \cdot \ldots \cdot (1 + a_k) \succeq 1 \geq 2^{1-D(\Gamma)} \cdot 2^k$$

and the assertion is true.

Assume it is true for the number of factors less than k, where $k > D(\Gamma) - 1$. Hence $k \geq D(\Gamma)$. By the definition of the Davenport constant we may assume, without loss of generality, that

$$a_1 \cdot \ldots \cdot a_t = 1, \text{ for some } 1 \leq t \leq D(\Gamma).$$

By the inductive assumption

$$\prod_{i=2}^{t}(1 + a_i^{-1}) \prod_{i=t+1}^{k}(1 + a_i) \succeq 2^{1-D(\Gamma)} \cdot 2^{k-1},$$

$$\prod_{i=2}^{k}(1 + a_i) \succeq 2^{1-D(\Gamma)} \cdot 2^{k-1}.$$

Hence

$$\prod_{i=1}^{k}(1+a_i) = \prod_{i=2}^{k}(1+a_i) + a_1\prod_{i=2}^{k}(1+a_i)$$

$$= \prod_{i=2}^{k}(1+a_i) + a_1 a_2 \cdot \ldots \cdot a_t \prod_{i=2}^{t}(1+a_i^{-1}) \prod_{i=t+1}^{k}(1+a_i)$$

$$= \prod_{i=2}^{k}(1+a_i) + \prod_{i=2}^{t}(1+a_i^{-1}) \prod_{i=t+1}^{k}(1+a_i) \succeq 2^{1-D(\Gamma)} \cdot 2^{k-1} + 2^{1-D(\Gamma)} \cdot 2^{k-1} = 2^{1-D(\Gamma)} \cdot 2^{k}.$$

By Lemma 2.5. it suffices to prove:

Theorem

For every finite Abelian group Γ, *for all* $a_1, \ldots, a_k \in \Gamma$, *and for all positive integers* $b_1, \ldots, b_k$ *we have*

$$\prod_{i=1}^{k}(1 + a_i + \ldots + {a_i}^{b_i}) \succeq 2^{1-D(\Gamma)} \prod_{i=1}^{k}(b_i + 1),$$

where $D(\Gamma)$ *is the Davenport constant of the group* Γ.

Proof. We use the identity (3) to get

$$\prod_{i=1}^{k}(1+a_i+\ldots+{a_i}^{b_i}) = \prod_{i=1}^{k}\sum_{j=0}^{b_i} 2^{j-b_i-1}(1+{a_i}^{j})(1+a_i)^{b_i-j} \tag{6}$$

$$= \sum_{\substack{0\le j_1\le b_1\\ 0\le j_2\le b_2\\ \vdots\\ 0\le j_k\le b_k}} \prod_{i=1}^{k} 2^{j_i-b_i-1}(1+{a_i}^{j_i})(1+a_i)^{b_i-j_i}.$$

By Lemma 2.6. we obtain

$$\sum_{\substack{0\le j_1\le b_1\\ 0\le j_2\le b_2\\ \vdots\\ 0\le j_k\le b_k}} \prod_{i=1}^{k} 2^{j_i-b_i-1}(1+a_i{}^{j_i})(1+a_i)^{b_i-j_i}$$

$$\succeq 2^{1-D(\Gamma)} \sum_{\substack{0\le j_1\le b_1\\ 0\le j_2\le b_2\\ \vdots\\ 0\le j_k\le b_k}} \prod_{i=1}^{k} 2^{j_i-b_i-1}2^{1+b_i-j_i} = 2^{1-D(\Gamma)} \sum_{\substack{0\le j_1\le b_1\\ 0\le j_2\le b_2\\ \vdots\\ 0\le j_k\le b_k}} 1$$

$$= 2^{1-D(\Gamma)} \prod_{i=1}^{k}(b_i+1).$$

Thus

$$\prod_{i=1}^{k}(1+a_i+\ldots+{a_i}^{b_i}) \succeq 2^{1-D(\Gamma)}\prod_{i=1}^{k}(b_i+1).$$

We have proved in [9] the following two statements.

Theorem 3.1.

For every finite Abelian group Γ, for all $g, a_1, \ldots, a_k \in \Gamma$, if there exists a solution of the equation $\sum_{i=1}^{k} a_i x_i = g$ in non-negative integers $x_i \leq b_i$, where b_i are positive integers, then the number of such solutions is at least

$$3^{1-D(\Gamma)} \prod_{i=1}^{k} (b_i + 1). \tag{7}$$

Remark 3.2.

Let $\Gamma = n\mathbb{Z}_2$ be a direct product of n cyclic groups of order two, $a_1, \dots, a_n$ a basis for Γ. Then the number of solutions of the equation

$$\sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n} a_i$$

in non-negative integers $x_i \leq b_i = 2$, equals 1.
Since $D(\Gamma) = n + 1$ (see Olson [2]) and $1 = 3^{1-D(\Gamma)} \prod_{i=1}^{n}(2+1)$,
we get that $3^{1-D(\Gamma)}$ is the best possible coefficient independent of a_i, b_i, g and dependent only on Γ.

Theorem 3.3.

For every finite Abelian group Γ, for all $g, a_1, \ldots, a_k \in \Gamma$, if there exists a solution of the equation $\sum_{i=1}^{k} a_i x_i = g$ in non-negative integers $x_i \leq b_i$, where $b_i \in \{2^s - 1 : s \in \mathbb{N}\}$, then the number of such solutions is at least

$$2^{1-D(\Gamma)} \prod_{i=1}^{k} (b_i + 1). \tag{8}$$

Lemma 3.4.

For every finite Abelian group Γ with multiplicative notation and for all $a_1, \ldots, a_k, g \in \Gamma$, the number of solutions of the equation $\prod_{i=1}^{k} {a_i}^{x_i} = g$ in non-negative integers $x_i \leq b_i$ is equal to N_1, where

$$g^{-1} \prod_{i=1}^{k} (1 + a_i + \ldots + {a_i}^{b_i}) = \sum_{h \in \Gamma} N_h h,$$

is an identity in $\mathbb{Q}[\Gamma]$.

Proof. We interpret the equation $g^{-1}\prod_{i=1}^{k}(1+a_i+\ldots+{a_i}^{b_i})=\sum_{h\in\Gamma}N_h h$ combinatorially. For $g\in\Gamma$ look at all sequences ${a_1}^{x_1}, {a_2}^{x_2}, \ldots, {a_k}^{x_k}$, that have product g, where $x_i\le b_i$ are non-negative integers. Then N_1 count those sequences. Therefore the number of solutions of the equation $\prod_{i=1}^{k}{a_i}^{x_i}=g$ in non-negative integers $x_i\le b_i$ is equal to N_1.

Lemma 3.5.

Theorem 3.1. with multiplicative notation is equivalent to the statement: for every finite Abelian group Γ, for all $g, a_1, \ldots, a_k \in \Gamma$, if there exists a solution of the equation $\prod_{i=1}^{k} {a_i}^{x_i} = g$ in non-negative integers $x_i \leq b_i$, where b_i are positive integers, then we have:

$$g^{-1} \prod_{i=1}^{k} (1 + a_i + \ldots + {a_i}^{b_i}) \succeq 3^{1-D(\Gamma)} \prod_{i=1}^{k} (b_i + 1), \tag{9}$$

where $D(\Gamma)$ is the Davenport constant of the group Γ.
Proof. This follows from Lemma 3.4 and Definition 2.4.

Lemma 3.6.

Theorem 3.3. with multiplicative notation is equivalent to the statement: for every finite Abelian group Γ, for all $g, a_1, \ldots, a_k \in \Gamma$, and for all positive integers $b_1, b_2, \ldots, b_k \in \{2^s - 1 : s \in \mathbb{N}\}$, if there exists a solution of the equation $\prod_{i=1}^{k} {a_i}^{x_i} = g$ in non-negative integers $x_i \leq b_i$, then we have relation:

$$g^{-1}\prod_{i=1}^{k}(1 + a_i + \ldots + {a_i}^{b_i}) \succeq 2^{1-D(\Gamma)}\prod_{i=1}^{k}(b_i + 1). \tag{10}$$

Proof. This follows from Lemma 3.4 and Definition 2.4.

Lemma 3.7.

For every finite Abelian group Γ and for all $g, a_1, a_2, \dots, a_k \in \Gamma$, if there exists a solution of the equation $\prod_{i=1}^{k} {a_i}^{x_i} = g$ in non-negative integers $x_i \leq 1$, then

$$g^{-1} \prod_{i=1}^{k} (1 + a_i) \succeq 2^{1-D(\Gamma)} \cdot 2^k. \tag{11}$$

Proof. We may assume that $\prod_{i=1}^{t} a_i = g$, where $1 \le t \le k$.

We have the identities

$$g^{-1}\prod_{i=1}^{k}(1+a_i) = g^{-1}\prod_{i=1}^{t} a_i \prod_{i=1}^{t}(1+a_i^{-1}) \prod_{i=t+1}^{k}(1+a_i) = \prod_{i=1}^{t}(1+a_i^{-1}) \prod_{i=t+1}^{k}(1+a_i).$$

By Theorem 1.1

$$\prod_{i=1}^{t}(1+a_i^{-1}) \prod_{i=t+1}^{k}(1+a_i) \succeq 2^{1-D(\Gamma)}2^k.$$

This implies

$$g^{-1}\prod_{i=1}^{k}(1+a_i) \succeq 2^{1-D(\Gamma)}2^k.$$

Lemma 3.8.

If $0 \leq t < b$, where t, b are integers, then $b - t + 1 \geq (\frac{2}{3})^t (b+1)$.

Proof. We verify by differentiation that the function $f(x) = 2(\frac{3}{2})^x - x - 2$ is increasing in the interval $(1, \infty)$. Since $f(0) = f(1) = 0, f(2) = \frac{1}{2}$ we get $2(\frac{3}{2})^t \geq t + 2$ for non-negative integers t. Hence $1 - \frac{t}{b+1} \geq 1 - \frac{t}{t+2} \geq (\frac{2}{3})^t$, and thus $b - t + 1 \geq (\frac{2}{3})^t (b+1)$.

Lemma 3.9.

For $s \geq 1$ we have the following identity in $\mathbb{Q}[\Gamma]$:

$$1 + x + x^2 + \ldots + x^{2^s - 1} = \prod_{j=1}^{s} (1 + x^{2^{j-1}}). \tag{12}$$

Proof. We proceed by induction on s.

Proof of Theorem 3.1.

We may find $0 \leq t_i \leq b_i$, where $1 \leq i \leq k$, such that $a_1{}^{t_1} a_2{}^{t_2} \cdot \ldots \cdot a_k{}^{t_k} = g$. By definition of the Davenport constant we may assume that

$$\sum_{i=1}^{k} t_i \leq D(\Gamma) - 1. \tag{13}$$

Let $t_i = b_i$ for $1 \leq i \leq s \leq k$; $t_i < b_i$ for $s+1 \leq i \leq k$; if $t_i < b_i$ for $1 \leq i \leq k$, then we take $s = 0$.

We have the identities

$$g^{-1}\prod_{i=1}^{s}(1+a_i+\ldots+a_i^{b_i})\prod_{i=s+1}^{k}({a_i}^{t_i}+{a_i}^{t_i+1}+\ldots+a_i^{b_i})=$$

$$=\Big(\Big(\prod_{i=1}^{s}a_i^{b_i}\Big)\Big(\prod_{i=s+1}^{k}a_i^{t_i}\Big)\Big)^{-1}\prod_{i=1}^{s}(1+a_i+\ldots+a_i^{b_i})\prod_{i=s+1}^{k}({a_i}^{t_i}+{a_i}^{t_i+1}+\ldots+a_i^{b_i})=$$

$$=\prod_{i=1}^{s}(1+a_i^{-1}+\ldots+(a_i^{-1})^{b_i})\prod_{i=s+1}^{k}(1+a_i+\ldots+{a_i}^{b_i-t_i}).$$

By Theorem 1.1.

$$\prod_{i=1}^{s}(1 + a_i^{-1} + \ldots + (a_i^{-1})^{b_i}) \prod_{i=s+1}^{k}(1 + a_i + \ldots + a_i{}^{b_i - t_i})$$

$$\succeq 2^{1-D(\Gamma)}\Big(\prod_{i=1}^{s}(b_i+1)\Big)\Big(\prod_{i=s+1}^{k}(b_i - t_i + 1)\Big).$$

We have by Lemma 3.8. that

$$2^{1-D(\Gamma)}\Big(\prod_{i=1}^{s}(b_i+1)\Big)\Big(\prod_{i=s+1}^{k}(b_i-t_i+1)\Big)$$

$$\geq 2^{1-D(\Gamma)}\Big(\prod_{i=1}^{s}(b_i+1)\Big)\Big(\prod_{i=s+1}^{k}(\tfrac{2}{3})^{t_i}(b_i+1)\Big)=$$

$$=2^{1-D(\Gamma)}(\tfrac{2}{3})^{\sum_{i=s+1}^{k}t_i}\prod_{i=1}^{k}(b_i+1)\geq 2^{1-D(\Gamma)}(\tfrac{2}{3})^{\sum_{i=1}^{k}t_i}\prod_{i=1}^{k}(b_i+1).$$

Since (13) it follows that

$$2^{1-D(\Gamma)}\left(\tfrac{2}{3}\right)^{\sum_{i=1}^{k} t_i}\prod_{i=1}^{k}(b_i+1) \geq 2^{1-D(\Gamma)}\left(\tfrac{2}{3}\right)^{D(\Gamma)-1}\prod_{i=1}^{k}(b_i+1) = 3^{1-D(\Gamma)}\prod_{i=1}^{k}(b_i+1).$$

Hence

$$g^{-1}\prod_{i=1}^{s}(1+a_i+\ldots+a_i^{b_i})\prod_{i=s+1}^{k}({a_i}^{t_i}+{a_i}^{t_i+1}+\ldots+a_i^{b_i}) \succeq 3^{1-D(\Gamma)}\prod_{i=1}^{k}(b_i+1).$$

Finally

$$g^{-1}\prod_{i=1}^{k}(1+a_i+\ldots+a_i^{b_i}) \succeq 3^{1-D(\Gamma)}\prod_{i=1}^{k}(b_i+1).$$

Proof of Theorem 3.3.

Let $b_i = 2^{s_i} - 1$, where $s_i \in \mathbb{N}$.
We take $0 \leq t_i \leq b_i$, where $1 \leq i \leq k$ such that $a_1^{t_1} a_2^{t_2} \cdot \ldots \cdot a_k^{t_k} = g$.
Since $0 \leq t_i \leq 2^{s_i} - 1$ we may find $\epsilon_{ji} \in \{0, 1\}$ such that

$$t_i = \sum_{j=1}^{s_i} \epsilon_{ji} 2^{j-1}$$

for $1 \leq i \leq k$.

Using (12) we obtain

$$a_i^{-t_i}(1 + a_i + \ldots + a_i{}^{b_i}) = a_i^{-t_i}\prod_{j=1}^{s_i}(1 + a_i^{2^{j-1}}) =$$

$$= a_i^{-\sum_{j=1}^{s_i}\epsilon_{ji}2^{j-1}}\prod_{j=1}^{s_i}(1+a_i^{2^{j-1}}) = \prod_{j=1}^{s_i} a_i^{-\epsilon_{ji}2^{j-1}}\prod_{j=1}^{s_i}(1+a_i^{2^{j-1}}) = \prod_{j=1}^{s_i} a_i^{-\epsilon_{ji}2^{j-1}}(1+a_i^{2^{j-1}}) =$$

$$= \prod_{j=1}^{s_i}(a_i^{-\epsilon_{ji}2^{j-1}} + a_i^{(1-\epsilon_{ji})2^{j-1}}) = \prod_{j=1}^{s_i}(1 + a_i^{\eta_{ji}2^{j-1}}),$$

where $\eta_{ji} = 1 - 2\epsilon_{ji} \in \{-1, 1\}$.

Thus

$$g^{-1}\prod_{i=1}^{k}(1+a_i+\ldots+{a_i}^{b_i})=\prod_{i=1}^{k}a_i^{-t_i}(1+a_i+\ldots+{a_i}^{b_i})=\prod_{i=1}^{k}\prod_{j=1}^{s_i}(1+a_i^{\eta_{ji}2^{j-1}}).$$

By Theorem 1.1.

$$\prod_{i=1}^{k}\prod_{j=1}^{s_i}(1+a_i^{\eta_{ji}2^{j-1}}) \succeq 2^{1-D(\Gamma)}\prod_{i=1}^{k}\prod_{j=1}^{s_i}2 = 2^{1-D(\Gamma)}\prod_{i=1}^{k}2^{s_i} = 2^{1-D(\Gamma)}\prod_{i=1}^{k}(b_i+1),$$

which implies

$$g^{-1}\prod_{i=1}^{k}(1+a_i+\ldots+a_i{}^{b_i}) \succeq 2^{1-D(\Gamma)}\prod_{i=1}^{k}(b_i+1).$$

[1.] Karol CWALINA and Tomasz SCHOEN, *The number of solutions of a homogeneous linear congruence*, Acta Arith. 153 (2012), pp. 271-279.

[2.] John E. OLSON, *A Combinatorial Problem on Finite Abelian Groups,I.* J. Number Theory 1, (1969), pp. 8-10.

[3.] John E. OLSON, *A Combinatorial Problem on Finite Abelian Groups,II.* J. Number Theory 1, (1969), pp. 195-199.

[4.] Andrzej SCHINZEL, *The number of solutions of a linear homogeneous congruence.* Diophantine Approximation: Festschrift for Wolfgang Schmidt (H.-P.Schlickewei, K. Schmidt, R.F. Tichy, eds.), pp. 363-370 (Developments in Mathematics 16, Springer-Verlag, 2008).

[5.] Andrzej SCHINZEL, *The number of solutions of a linear homogeneous congruence II.* In: Chen,W., Gowers, T., Halberstam, H., Schmidt, W., Vaughan, R.C. (eds.) Analytical Number Theory: Essays in Honour of Klaus F. Roth. Cambridge University Press, 2009, pp. 402-413, with appendix by Jerzy KACZOROWSKI.

[6.] Andrzej SCHINZEL and Maciej ZAKARCZEMNY, *On a linear homogeneous congruence.*, Colloq. Math., 106 (2006), pp. 283-292.

[7.] Michael DRMOTA i Mariusz SKAŁBA, *Equidistribution of Divisors modulo m.* Preprint.(1997).

[8.] Maciej ZAKARCZEMNY, *Number of solutions in a box of a linear homogeneous equation in an Abelian group*, Acta Arith. 155 (2012), pp. 227-231.

[9.] Maciej ZAKARCZEMNY, *Number of solutions in a box of a linear equation in an Abelian group*, (to appear).

Thank you for your attention.

Theorem 1.1 we may rewrite in the form:
for all positive integers $n_1 \mid n_2 \mid \ldots \mid n_l$, b_i and for all integers a_{ij}, where $1 \leq i \leq k$, $1 \leq j \leq l$ the number of solutions of the system

$$\begin{cases} a_{11}x_1 + a_{21}x_2 + \ldots + a_{k1}x_k \equiv 0 \pmod{n_1}, \\ a_{12}x_1 + a_{22}x_2 + \ldots + a_{k2}x_k \equiv 0 \pmod{n_2}, \\ \vdots \\ a_{1l}x_1 + a_{2l}x_2 + \ldots + a_{kl}x_k \equiv 0 \pmod{n_l}, \end{cases}$$

in non-negative integers $x_i \leq b_i$ is at least

$$2^{1-D(\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \ldots \oplus \mathbb{Z}_{n_l})} \prod_{i=1}^{k} (b_i + 1).$$

Known Davenports constants

1. $D(\mathbb{Z}_n) = n$.
2. If $1 < m|n$ then $D(\mathbb{Z}_m \oplus \mathbb{Z}_n) = m + n - 1$.
3. If $G = \mathbb{Z}_{p^{e_1}} \oplus \mathbb{Z}_{p^{e_2}} \oplus \ldots \oplus \mathbb{Z}_{p^{e_r}}$ a finite $p-group$ then $D(G) = 1 + \sum_{i=1}^{r}(p^{e^i} - 1)$.
4. If G is a finite abelian group , then there exist uniquely determined integers $1 < d_1|d_2| \ldots |d_r$, such that $G \cong \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \ldots \oplus \mathbb{Z}_{d_r}$. We obtain a sequence of $\sum_{i=1}^{r}(d_i - 1)$, hence $D(G) \geq 1 + \sum_{i=1}^{r}(d_i - 1)$.
5. It is unknown whether $D(G) = 1 + \sum_{i=1}^{r}(d_i - 1)$ holds true for all groups of rank $r = 3$.
6. $D(\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{3d}) = 3d + 4$.
7. Currently (2008) the best upper bound for $D(G)$ is due to Van Emde Boas and Kruyswijk and Meshulam: $D(G) \leq n + \left[n \log \frac{|G|}{n}\right]$, where n is the maximum possible order of an element also known as the exponent of the group.
8. $D(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{2n}) = 2n + 3$, with odd n.
9. $D(\mathbb{Z}_m \oplus \mathbb{Z}_n \oplus \mathbb{Z}_n \oplus \mathbb{Z}_{2n}) > 1 + (m-1) + (n-1) + (n-1) + (2n-1)$, for every odd n, m with $m \geq 3$ and $m|n$.

Known Davenports constants of non-abelian finite groups

1. Dihedral groups. If $D_{2n} = \langle x, y : x^2 = y^n = 1, \ yx = xy^{-1} \rangle$, then $D(D_{2n}) = n + 1$,
2. Dicyclic groups. If $Q_{4n} = \langle x, y : x^2 = y^n, \ y^{2n} = 1, yx = xy^{-1} \rangle$, then $D(Q_{4n}) = 2n + 1$,
3. A non-abelian group of order pq exists only when $p | q - 1$, and such a group is unique.
   $G_{pq} = \langle x, y : x^p = y^q = 1, yx = xy^s \rangle$, where $s^p \equiv 1 \mod q$, $s \not\equiv 1 \mod q$.
   We have $D(G_{pq}) = p + q - 1$.

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Group ring

Group ring $\mathbb{Q}[\Gamma]$ is a $\mathbb{Q}$-vector space with basis Γ and with multiplication defined distributively using the given multiplication of Γ.

$$\left(\sum_{g\in\Gamma}\alpha_g g\right)\cdot\left(\sum_{g\in\Gamma}\beta_g g\right)=\sum_{x\in\Gamma}\left(\sum_{gh=x}\alpha_g\beta_h\right)x.$$

We have $\sum\limits_{g\in\Gamma}\alpha_g g=\sum\limits_{g\in\Gamma}\beta_g g$ iff $\alpha_g=\beta_g$ for all $g\in\Gamma$.
Instead $\sum\limits_{g\in\Gamma}0g$ we write 0.
Instead $1g$ we write g.
Instead $(-\alpha)g$ we write $-\alpha g$.
We denoting the group unit 1_Γ and the unit element of the ring $\mathbb{Q}$ by the same symbol 1.
We denoting addition operation in $\mathbb{Q}[\Gamma]$ and in $\mathbb{Q}$ by the same symbol.

If $1_\Gamma=1$, then the additive group of $\mathbb{Q}[\Gamma]$ becomes an extension of the additive group of $\mathbb{Q}$, thus the use of the same symbol $+$ is legitimate.