EXERCISES 3

Exercises

1. Compute $(935, 1122)$.

2. Compute $(168, 252, 294)$.

3. Find integers x and y such that $13x + 15y = 1$.

4. Construct four relatively prime integers a, b, c, d such that no three of them are relatively prime.

5. Prove that $(n, n+2) = 1$ is n is odd and $(n, n+2) = 2$ is n is even.

6. Prove that $2n+5$ and $3n+7$ are relatively prime for every integer n.

7. Prove that $3n+2$ and $5n+3$ are relatively prime for every integer n.

8. Prove that $n!+1$ and $(n+1)!+1$ are relatively prime for every positive integer n.

9. Let a, b, and d be positive integers. Prove that if $(a, b) = 1$ and d divides a, then $(d, b) = 1$.

10. Let a and b be positive integers. Prove that $(a, b) = a$ if and only if a divides b.

11. Let a, b, c, d be integers such that $ad - bc = 1$. For integers u and v, define
$$u' = au + bv$$
and
$$v' = cu + dv.$$
Prove that $(u, v) = (u', v')$.

Hint: Express u and v in terms of u' and v'.

12. Let $G = \{2\mathbf{Z}, 1+2\mathbf{Z}\}$, where $2\mathbf{Z}$ denotes the set of even integers and $1+2\mathbf{Z}$ the set of odd integers. Define addition of elements of G by
$$2\mathbf{Z} + 2\mathbf{Z} = (1+2\mathbf{Z}) + (1+2\mathbf{Z}) = 2\mathbf{Z}$$
and
$$2\mathbf{Z} + (1+2\mathbf{Z}) = (1+2\mathbf{Z}) + 2\mathbf{Z} = 1+2\mathbf{Z}.$$
Prove that G is an additive abelian group.

13. Use the Euclidean algorithm to compute the greatest common divisor of 35 and 91, and to express $(35, 91)$ as a linear combination of 35 and 91. Compute the simple continued fraction for $91/35$.

14. Use the Euclidean algorithm to write the greatest common divisor of 4534 and 1876 as a linear combination of 4534 and 1876. Compute the simple continued fraction for $4534/1876$.

15. Use the Euclidean algorithm to compute the greatest common divisor of 1197 and 14280, and to express $(1197, 14280)$ as a linear combination of 1197 and 14280.

Solutions

Exercise 1, solution.

We have $1122 = 1 \cdot 935 + 187$, $935 = 5 \cdot 187$. Hence $(1122, 935) = 187$.

Exercise 2, solution 1.

We have $252 = 1 \cdot 168 + 84$, $168 = 2 \cdot 84$. Hence $(252, 168) = 84$.

We have $294 = 3 \cdot 84 + 42$, $84 = 2 \cdot 42$. Hence $(294, 84) = 42$.

Note that $(168,252,294)|(252,168)$. Thus $(168,252,294)|(294,84)$, $(168,252,294)|42$.
On the other hand $42|(168,252,294)$, therefore we get equality $42=(168,252,294)$. Exercise 2, solution 2.
We have factorization into prime numbers

$$168=2^3\cdot 3\cdot 7,\ 252=2^2\cdot 3^2\cdot 7,\ 294=2\cdot 3\cdot 7^2.$$

Hence $(168,252,294)=2\cdot 3\cdot 7$.
Exercise 3, solution.
We have $15=1\cdot 13+2$, $13=6\cdot 2+1$.
Hence $2=15-1\cdot 13$, and then substitute the result into the other equation $1=13-6\cdot 2$, we get

$$1=13-6(15-1\cdot 13)=7\cdot 13-6\cdot 15.$$

Thus $x=7$, $y=-6$.
Exercise 6, solution.
We have $(2n+5,3n+7)=1$, since $3(2n+5)-2(3n+7)=1$.
Exercise 7, solution.
We have $(3n+2,5n+3)=1$, since $5(3n+2)-3(5n+3)=1$.
Exercise 8, solution.
By definition of the greatest common divisor $(n!+1,(n+1)!+1)|(n!+1)$.
We have $(n!+1,(n+1)!+1)|n$, since $(n+1)(n!+1)-1((n+1)!+1)=n$.
But$(n,n!+1)=1$, hence $(n!+1,(n+1)!+1)=1$.
Exercise 11, solution.

$$\begin{aligned} u' &= au+bv \\ v' &= cu+dv. \end{aligned} \quad \Rightarrow \quad \begin{aligned} u &= du'-bv' \\ v &= -cu'+av' \end{aligned}$$

Exercise 13,14,15, solution.

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cfdisrep(cf(91/35));
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$$2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2}}}$$

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gcd(91,35);
7
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cfdisrep(cf(4534/1876));
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$$2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{38+\cfrac{1}{2}}}}}}$$

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gcd(4534,1876);
2
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cfdisrep(cf(14280/1197));
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$$11+\cfrac{1}{1+\cfrac{1}{13+\cfrac{1}{4}}}$$

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gcd(14280,1197);
21
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