Exercises 10

1. Find an integer g that is a primitive root modulo 5^k for all $k \geq 1$. Find a primitive root modulo 10. Find a primitive root modulo 50.
2. For $k \geq 1$, let e_k be the order of 5 modulo 3^k. Prove that
$$e_k = 2 \cdot 3^{k-1}.$$
3. Prove that p divides the binomial coefficient $\binom{p}{i}$ for $i = 1, 2, \ldots, p-1$.
4. Prove that if g is a primitive root modulo p^2, then g is a primitive root modulo p^k for all $k \geq 2$.
5. Let p be an odd prime. Prove that
$$(1+px)^{p^k} \equiv 1 + p^{k+1}x \pmod{p^{k+2}}$$
for every integer x and every nonnegative integer k.
6. (Wagstaff [151]) Let p be an odd prime, and let $a \neq \pm 1$ be an integer not divisible by p. Let d be the order of a modulo p, and let k_0 be the largest integer such that $a^d \equiv 1 \pmod{p^{k_0}}$. Prove that if $k \geq k_0$ is a solution of the exponential congruence
$$a^k \equiv 1 \pmod{p^k}, \tag{3.6}$$
then
$$\frac{p^k}{k} < \frac{a^d}{d},$$
and so congruence (3.6) has only finitely many solutions.

   Hint: Apply Theorem 3.6.
7. Use Exercise 6 to prove that the exponential congruence
$$9^k \equiv 1 \pmod{7^k} \text{ has no solutions.}$$
8. Find all solutions of the exponential congruence $17^k \equiv 1 \pmod{15^k}$.
9. Find all solutions of the exponential congruence $3^k \equiv 1 \pmod{2^k}$.
10. Let $\{x\}$ denote the fractional part of x. Compute $\left\{ \left(\frac{3}{2}\right)^n \right\}$ for $n = 1, \ldots, 10$. Let r_n be the least nonnegative residue of 3^n modulo 2^n. Show that
$$\left\{ \left(\frac{3}{2}\right)^n \right\} = \frac{r_n}{3^n}.$$

    Remark. It is an important unsolved problem in number theory to understand the distribution of the fractional parts of the powers of 3/2 in the interval [0, 1).