Exercises 7

1. Find all solutions of the system of congruences

$$\begin{aligned} x &\equiv 4 \pmod{5}, \\ x &\equiv 5 \pmod{6}. \end{aligned}$$

2. Find all solutions of the system of congruences

$$\begin{aligned} x &\equiv 5 \pmod{12}, \\ x &\equiv 8 \pmod{9}. \end{aligned}$$

3. Find all solutions of the system of congruences

$$\begin{aligned} x &\equiv 5 \pmod{12}, \\ x &\equiv 8 \pmod{10}. \end{aligned}$$

4. Find all solutions of the system of congruences

$$\begin{aligned} 2x &\equiv 1 \pmod{5}, \\ 3x &\equiv 4 \pmod{7}. \end{aligned}$$

5. Find all integers that have a remainder of 1 when divided by 3, 5, and 7.

6. Find all integers that have a remainder of 2 when divided by 4 and that have a remainder of 3 when divided by 5.

7. Find all solutions of the congruence

$$f(x) = 5x^3 - 93 \equiv 0 \pmod{231}.$$

8. (Bhaskara, sixth century) A basket contains n eggs. If the eggs are removed $2, 3, 4, 5$, or 6 at a time, then the number of eggs that remain in the basket is $1, 2, 3, 4$, or 5, respectively. If the eggs are removed 7 at a time, then no eggs remain. What is the smallest number n of eggs that could have been in the basket at the start of this procedure?

   Hint: The first condition implies that $n \equiv 1 \pmod{2}$.

9. Let f be a polynomial with integer coefficients. For $m \geq 1$, let $N_f(m)$ denote the number of pairwise incongruent solutions of $f(x) \equiv 0 \pmod{m}$. Prove that the function $N_f(m)$ is multiplicative, that is, $N_f(m_1 m_2) = N_f(m_1)N_f(m_2)$ if $(m_1, m_2) = 1$.

10. Let $m_1, \ldots, m_k$ be pairwise relatively prime positive integers and $m = m_1 \cdots m_k$. Define the map

$$f : (\mathbf{Z}/m\mathbf{Z})^\times \to (\mathbf{Z}/m_1\mathbf{Z})^\times \times \cdots \times (\mathbf{Z}/m_k\mathbf{Z})^\times$$

   by

$$f(a + m\mathbf{Z}) = (a + m_1\mathbf{Z}, \ldots, a + m_k\mathbf{Z}).$$

   Use the Chinese remainder theorem to show directly that this map is one-to-one and onto.