EXERCISE 4

1. Compute the simple continued fraction $\langle 2,1,2,1,1,4\rangle$ to 4 decimal places, and compare this number to e.

2. Prove that
$$\langle a_0, a_1, \ldots, a_N\rangle = a_0 + \frac{1}{\langle a_1, \ldots, a_N\rangle}.$$

3. Let $N \geq 1$. Prove that
$$\langle a_0, a_1, \ldots, a_{N-2}, a_{N-1}, 1\rangle = \langle a_0, a_1, \ldots, a_{N-2}, a_{N-1}+1\rangle.$$

4. Let $x = \langle a_0, a_1, \ldots, a_N\rangle$ be a finite simple continued fraction whose partial quotients a_i are integers, with $N \geq 1$ and $a_N \geq 2$. Let $[x]$ denote the integer part of x and $\{x\}$ the fractional part of x. Prove that
$$[x] = a_0$$
and
$$\{x\} = \frac{1}{\langle a_1, \ldots, a_N\rangle}.$$

5. Let $\frac{a}{b}$ be a rational number that is not an integer. Prove that there exist unique integers $a_0, a_1, \ldots, a_N$ such that $a_i \geq 1$ for $i = 1, \ldots, N-1$, $a_N \geq 2$, and
$$\frac{a}{b} = \langle a_0, a_1, \ldots, a_{N-1}, a_N\rangle.$$

Hint: By Exercise 7, if
$$x = \langle a_0, a_1, \ldots, a_N\rangle = \langle b_0, b_1, \ldots, b_M\rangle$$
with $a_i, b_j \in \mathbf{Z}$ and $a_N, b_M \geq 2$, then $a_0 = [x] = b_0$.

6. Prove that
$$\langle a_0, a_1, \ldots, a_N, a_{N+1}\rangle = \langle a_0, a_1, \ldots, a_N + \frac{1}{a_{N+1}}\rangle.$$

7. Let $\langle a_0, a_1, \ldots, a_N\rangle$ be a finite simple continued fraction. Define
$$p_0 = a_0,$$
$$p_1 = a_1a_0 + 1,$$
and
$$p_n = a_np_{n-1} + p_{n-2} \qquad \text{for } n = 2, \ldots, N.$$
Define
$$q_0 = 1,$$
$$q_1 = a_1,$$
and
$$q_n = a_nq_{n-1} + q_{n-2} \qquad \text{for } n = 2, \ldots, N.$$
Prove that
$$\langle a_0, a_1, \ldots, a_n\rangle = \frac{p_n}{q_n}$$
for $n = 0, 1, \ldots, N$. The continued fraction $\langle a_0, a_1, \ldots, a_n\rangle$ is called the nth *convergent* of the continued fraction $\langle a_0, a_1, \ldots, a_N\rangle$.

8. Compute the convergents p_n/q_n of the simple continued fraction $\langle 1,2,2,2,2,2,2\rangle$. Compute p_6/q_6 to 5 decimal places, and compare this number to $\sqrt{2}$.

9. Let $\langle a_0, a_1, \ldots, a_N\rangle$ be a finite simple continued fraction, and let p_n and q_n be the numbers defined in Exercise 10. Prove that
$$p_nq_{n-1} - p_{n-1}q_n = (-1)^{n-1}$$
and for $n = 1, \ldots, N$. Prove that if $a_i \in \mathbf{Z}$ for $i = 0, 1, \ldots, N$, then $(p_n, q_n) = 1$ for $n = 0, 1, \ldots, N$.

10. We define a sequence of integers as follows:
$$\begin{aligned} f_0 &= 0, \\ f_1 &= 1, \\ f_n &= f_{n-1} + f_{n-2} \qquad \text{for } n \geq 2. \end{aligned}$$
The integer f_n is called the nth *Fibonacci number*. Compute the Fibonacci numbers f_n for $n = 2, 3, \ldots, 12$. Prove that $(f_n, f_{n+1}) = 1$ for all nonnegative integers n.

11. Compute the convergents p_n/q_n of the simple continued fraction $\langle 1,1,1,1,1,1,1\rangle$. Observe that
$$\frac{p_n}{q_n} = \frac{f_{n+1}}{f_n}$$
for $n = 0, 1, \ldots, 6$.

12. Prove that
$$f_{n+1}f_{n-1} - f_n^2 = (-1)^n$$
for all positive integers n.

13. Compute the standard factorization of 15!.

14. Prove that $n, n+2, n+4$ are all primes if and only if $n = 3$.

15. Prove that $n, n+4, n+8$ are all primes if and only if $n = 3$.

16. Let $n \geq 2$. Prove that $(n+1)! + k$ is composite for $k = 2, \ldots, n+1$. This shows that there exist arbitrarily long intervals of composite numbers.

17. Prove that $n^5 - n$ is divisible by 30 for every integer n.

18. Find all primes p such that $29p + 1$ is a square.

19. The prime numbers p and q are called *twin primes* if $|p - q| = 2$. Let p and q be primes. Prove that $pq + 1$ is a square if and only if p and q are twin primes.

20. Prove that if p and q are twin primes greater than 3, then $p + q$ is divisible by 12.